Q1: let g, ∇ be induced metric, connection on M

take $p \in M$, $v \in T_pM$

let γ = g-geodesic s.t. $\gamma(0) = \bar{\gamma}(0) = p$, $\gamma'(0) = \bar{\gamma}'(0) = v$
$\bar{\gamma}$ = $\bar{g}$-geodesic

M tot. geodesic $\Rightarrow$ $\bar{\gamma}$ lies in M
(A)

$\Rightarrow \bar{\gamma}' \in T_{\bar{\gamma}}M$

$\Rightarrow \nabla_{\bar{\gamma}'}\bar{\gamma}' = \pi(\bar{\nabla}_{\bar{\gamma}'}\bar{\gamma}') = 0$

$\Rightarrow \bar{\gamma} = \gamma$ by uniqueness of ODEs

$\Rightarrow$ every g-geodesic = $\bar{g}$-geodesic (B)

since p, v arbitrary

spse $\gamma = \bar{\gamma}$ $\forall$ choices of p, v (B)

$\Rightarrow \nabla_{\gamma'(0)}\gamma'(0) = \bar{\nabla}_{\bar{\gamma}'(0)}\bar{\gamma}'(0)$

$\Rightarrow B|_p(v, v) = \bar{\nabla}_{\bar{\gamma}'(0)}\bar{\gamma}'(0) - \nabla_{\bar{\gamma}'(0)}\gamma'(0) = 0$ (C)

$\Rightarrow B = 0$ $\forall p, v$

spse $B \equiv 0 \Rightarrow \bar{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + B(\gamma', \gamma')$
(C)
$= 0$

$\Rightarrow \gamma = \bar{\gamma} \Rightarrow \bar{\gamma}$ lies in $M \Rightarrow M$ tot. geodesic
(A)

Q2 : catenoid : $r = \cosh(z) = \sqrt{x^2+y^2}$

↳ parametrization : $F(\theta, z) = (\cosh(z)\cos\theta, \cosh(z)\sin\theta, z)$

$\theta \in [0, 2\pi)$, $z \in \mathbb{R}$

$\Rightarrow \partial_\theta F = (-\cosh(z)\sin\theta, \cosh(z)\cos\theta, 0)$

$\partial_z F = (\sinh(z)\cos\theta, \sinh(z)\sin\theta, 1)$

$\Rightarrow g_{\theta\theta} = \partial_\theta F \cdot \partial_\theta F = \cosh^2(z)$

$g_{\theta z} = \partial_\theta F \cdot \partial_z F = 0$

$g_{zz} = \sinh^2(z) + 1 = \cosh^2(z)$

so $g_{ij} = \begin{array}{cc} \theta & z \\ \hline \cosh^2(z) & 0 \\ 0 & \cosh^2(z) \end{array}$

choice of unit normal : $\nu = \dfrac{(-\cos\theta, -\sin\theta, \sinh(z))}{\sqrt{1+\sinh^2(z)}}$

scalar second fundamental form in coords : $h_{ij} = \partial^2_{ij} F \cdot \nu$

$\partial^2_\theta F = (-\cosh(z)\cos\theta, -\cosh(z)\sin\theta, 0)$

$\partial^2_{\theta z} F = (-\sinh(z)\sin\theta, \sinh(z)\cos\theta, 0)$

$\partial^2_{zz} F = (\cosh(z)\cos\theta, \cosh(z)\sin\theta, 0)$

$\Rightarrow h_{\theta\theta} = \partial^2_\theta F \cdot \nu = \dfrac{\cosh(z)}{\sqrt{1+\sinh^2(z)}} = 1$, $h_{\theta z} = 0$, $h_{zz} = \dfrac{-\cosh(z)}{\sqrt{1+\sinh^2(z)}} = -1$

$$\Rightarrow \quad h^i{}_j = g^{ip} h_{pj} = \frac{1}{\cosh^2(z)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow$ H = scalar mean curvature

$= \mathrm{tr}_g h$

$= 0$

Q3: recall Gauss-Bonnet says: if Ω curved polygon in (M, g)

$$\Rightarrow \int_{\Omega} K + \int_{\partial\Omega} k + \sum \varepsilon_i = 2\pi \chi(\Omega)$$

Gauss curvature

geodesic curvature of ∂

exterior angles

Euler characteristic

if Ω geodesic triangle $\Rightarrow \chi = V - E + F$

$= 3 - 3 + 1$

$= 1$

and $\partial\Omega$ has zero curvature

and $\underbrace{\sum \varepsilon_i}_{\text{ext. angles}} = 3\pi - \underbrace{\sum \theta_i}_{\text{interior angles}}$

since $K = \text{const}$

$$\Rightarrow K|\Omega| + 3\pi - \sum \theta_i = 2\pi$$

$$\Rightarrow \sum \theta_i = \pi + K|\Omega|$$

Q4: given $p \in M$, $v \in N_p M$

choose coords $x^1 \ldots x^{n+k}$ of $\overline{M}$ near p s.t. $(x^i(0)) = p$

s.t. $M = \{ x^{n+1} = \ldots = x^{n+k} = 0 \}$ in these coords

choose $E_1 \ldots E_k$ = loc. OH frame for NM near p.

identify $T_{(p,v)}(NM) \cong T_pM \oplus N_pM \cong T_p\overline{M}$

via $a^1 \frac{\partial}{\partial x^1} + \ldots + a^n \frac{\partial}{\partial x^n} + b^1 \frac{\partial}{\partial v^1} + \ldots + b^k \frac{\partial}{\partial v^k}$

$$\longmapsto \left(a^i \frac{\partial}{\partial x^i}\Big|_p ,\ b^i E_i\big|_p \right) \quad \circledast$$

claim: $\exists$ neighborhood U_p of $(p,0)$ in $N_p M$. s.t.

the map $\exp : U_p \to \overline{M}$ = diffeo^m onto its image

$(p,v) \mapsto \exp_p(v)$

suffices to show $D\exp\big|_{(p,0)} : T_{(p,0)}(NM) \to T_p\overline{M}$ is non-singular

$\hookrightarrow$ identify $T_{(p,0)}(NM)$ w/ $T_p\overline{M}$ via $\circledast$

$$\Rightarrow \frac{\partial}{\partial x^i}\Big|_{\substack{x^i=0\\ v^i=0}} \exp_x(v) = \frac{\partial}{\partial x^i}(x^1, \ldots, x^n, 0, \ldots, 0) \quad \text{since } \exp_q(0) = q$$

$$= \frac{\partial}{\partial x^i} \quad \text{for } i = 1 \ldots n$$

and $\frac{\partial}{\partial v^j}\Big|_{\substack{x^i=0\\ v^i=0}} \exp_x(v) = D\exp_0\big|_0 (E_j) = E_j$ since $D\exp_q|_0 = id$

$j = 1 \ldots k$

so $D\exp\big|_{(p,0)} = Id$ under identification $\frac{\partial}{\partial x^i} \leftrightarrow \frac{\partial}{\partial x^i}$

$\frac{\partial}{\partial v^j} \leftrightarrow E_j$

$\Rightarrow$ non-singular

since M compact

$\Rightarrow \exists\, \varepsilon > 0$ st $\forall p \in M$

$\exp : B_\varepsilon(p,0) \cap NM \to \overline{M}$ = diffeo onto image

claim: $\exists\, \varepsilon' > 0$ st $\exp : (NM)_{\varepsilon'} \to \overline{M}$ = diffeo onto image

$\exp : (NM)_\varepsilon \to \overline{M}$ = loc. diffeo

suffices to show $\exp\big|_{(NM)_{\varepsilon'}}$ can be made injective for some $\varepsilon' \le \varepsilon$

spse $\nexists$ such ε'

$\hookrightarrow \exists\, p_i, \hat{p}_i \in M$ and $v_i \in N_{p_i}M$, $\hat{v}_i \in N_{\hat{p}_i}M$

with $|v_i| < 1/i$, $|\hat{v}_i| < 1/i$

st $\exp_{p_i}(v_i) = \exp_{\hat{p}_i}(\hat{v}_i)$ but $(p_i, v_i) \neq (\hat{p}_i, \hat{v}_i)$

since M cpct, wlog $p_i \to p \in M$

$\hookrightarrow d(p_i, \hat{p}_i) \le d(p_i, \exp_{p_i}(v_i)) + d(\hat{p}_i, \exp_{\hat{p}_i}(\hat{v}_i))$

$\le 2/i \to 0$

$\Rightarrow \hat{p}_i \to p$ also

$\Rightarrow (p_i, v_i), (\hat{p}_i, \hat{v}_i) \in B_\varepsilon(p,0) \cap NM \quad \forall i \gg 1$

$\Rightarrow (p_i, v_i) = (\hat{p}_i, \hat{v}_i)$ for $i \gg 1$ since $\exp\big|_{B_\varepsilon(p,0) \cap NM}$ = diffeo $\square$

$\lightning$

So $\exists\, \varepsilon > 0$ st $\exp : (NM)_\varepsilon \longrightarrow \bar{M}$ = diffeo onto image

need to show image $= M_\varepsilon$

since $d(p, \exp_p(v)) \leq |v| < \varepsilon$ for $(p,v) \in (NM)_\varepsilon$

$$\Rightarrow \exp\big((NM)_\varepsilon\big) \subset M_\varepsilon$$

since $D\exp_p|_0$ non-singular $\forall p$, and M cpct

$\Rightarrow$ can also assume that $\exp_p : B_\varepsilon(0) \subset T_p\bar{M} \longrightarrow \bar{M}$ diffeo onto image, $\forall p \in M$

given $q \in M_\varepsilon \Rightarrow \exists\, p \in M$ s.t $d(p,q) = d(p, M) < \varepsilon$ (again using cpctness of M)

$\Rightarrow \gamma(t) = \exp_p(tv)$ = min'z geodesic $p \longrightarrow q$ $(t \in [0,1])$ for some $v \in T_pM$

if $\gamma_s(t)$ any family of curves st $\gamma_s(0) \in M$, $\gamma_s(1) = q$ and $\gamma_0(t) = \gamma$

$\Rightarrow L\gamma_s \geq d(q, M) = L\gamma_0 \qquad V = \partial_s|_{s=0}$

$$\Rightarrow \quad 0 = \frac{d}{ds}\Big|_{s=0} L\gamma_s = \int_0^1 \langle V, \frac{D\gamma'}{dt}\rangle dt + \langle V, \gamma' \rangle \Big|_0^1$$

$$= -\langle V(0), \gamma'(0)\rangle$$

since $\gamma_s(0) \in M \Rightarrow V(0) \in T_pM$

taking $V(0) \in T_pM$ arbitrary $\Rightarrow \gamma'(0) \perp T_pM$

$$\Rightarrow \gamma'(0) = v \in N_pM$$

and $d(p,q) = L\gamma = |v|$ $< \varepsilon$

$$\Rightarrow (p,v) \in (NM)_\varepsilon$$

$$\Rightarrow \exp((NM)_\varepsilon) = M_\varepsilon$$

In fact we showed that $\operatorname{dist}(x, M) = |\exp^{-1}(x)| \quad \forall x \in M_\varepsilon$

inverse of $\exp : (NM)_\varepsilon \to M_\varepsilon$

$$\Rightarrow \operatorname{dist}(x,M)^2 = |\exp^{-1}(x)|^2$$

= smooth for $x \in M_\varepsilon$

$\operatorname{dist}(x,M)^2$ can fail to be smooth for x far from M

eg. if $\overline{M} = S^1 \times \mathbb{R}$, $M = \{1\} \times \mathbb{R}$

$$\Rightarrow \operatorname{dist}_{\overline{M}}((e^{i\theta}, x), (e^0, x)) = \min\{\theta, 2\pi - \theta\}$$

for $\theta \in [0, 2\pi)$