Math 60670 Homework 7

Due Tuesday, April 2.

Problem 1: A curve $\gamma : [0, b) \to M$ is said to converge to infinity if for every compact set K, there is a T so that $\gamma(t) \notin K$ for $t > T$ (i.e. γ converges to infinity in the one-point compactification of M at infinity). Show that a Riemannian manifold (M, g) is complete if and only if every regular curve that converges to infinity has infinite length.

Problem 2: Let ω be a 1-form on (M, g). Show that

$$(R(X,Y)\omega)(Z) := (\nabla^2_{X,Y}\omega)(Z) - (\nabla^2_{Y,X}\omega)(Z) = -\omega(R(X,Y)Z).$$

Problem 3: Prove the second Bianchi identity

$$(\nabla_T R)(X,Y,Z,W) + (\nabla_X R)(Y,T,Z,W) + (\nabla_Y R)(T,X,Z,W) = 0.$$

Hint: use normal coordinates.

Problem 4: Show the scalar curvature Scal at a point $p \in M$ can be written

$$\mathrm{Scal}(p) = \frac{n}{|S^{n-1}|}\int_{S^{n-1}} \mathrm{Ric}_p(\theta,\theta)\,d\theta,$$

where S^{n-1} is the Euclidean $(n-1)$-sphere, and $|S^{n-1}|$ is its $(n-1)$-volume.