Q2: let M be complete, spse $\gamma: [0,b) \to M$ converges to ∞

if $\underset{R}{\underset{\shortparallel}{L\gamma}} < \infty \Rightarrow \gamma(t) \subset B_R(\gamma(0)) \quad \forall t$

$\Rightarrow \gamma(t) \subset \overline{B_R(\gamma(0))} \quad \forall t$

and $\overline{B_R(\gamma(0))}$ cpct since closed + bounded $\lightning$

so we must have $L\gamma = \infty$

spse every curve in M converging to ∞ has infinite length

WTS: M complete

let $\gamma: I \to M$ be geodesic PBAL

spse γ defined for $t < t_*$, but cannot be extended past t_*

claim: $\gamma: [0, t_*) \to M$ converges to ∞

otherwise, $\exists$ cpct K and $t_i \to t_*$ st $\gamma(t_i) \in K$

$\Rightarrow$ (subseq) $\gamma(t_i) \to p \in K$ by compactness

(since M = metric space, compactness and seq. compactness are the same)

but then $d(\gamma(t), p) \leq |t - t_i| + d(\gamma(t_i), p) \quad \forall i$

$\to 0$ as $t \to t_*$

so $\gamma(t) \to p$

choose $\delta > 0$ st $\exp_q\big|_{B_\delta(0)}$ = diffeom $\forall q \in B_\delta(p)$

$\Rightarrow \gamma(t) = \exp_{\gamma(s)}((t-s)\gamma'(s))$ $\forall$ s near t_*

$\forall\ |t-s| < \delta$

$\Rightarrow \gamma$ can be extended to $t \le t_* + \delta$ $\lightning$

by hypothesis $\infty = L\gamma|_{[0,t_*)} = \int_0^{t_*} |\gamma'|\, dt$

$= t_*$ $\lightning$

$\hookrightarrow$ so γ must be defined $\forall t$

$\Rightarrow$ M complete

Q2 choose normal coords @ $p \Rightarrow \nabla \partial_i |_p = 0$

$$\Rightarrow (\nabla^2_{\partial_i, \partial_j} \omega)(\partial_k) = \partial_i \left[(\nabla_{\partial_j} \omega)(\partial_k) \right] - (\nabla_{\cancel{\nabla_{\partial_i} \partial_j}} \omega)(\partial_k) - (\nabla_{\partial_j} \omega)(\cancel{\nabla_{\partial_i} \partial_k})$$

$$= \partial_i \left[(\nabla_{\partial_j} \omega)(\partial_k) \right] \quad @ \; p$$

$\omega = \omega_i \, dx^i$

$$= \partial_i (\partial_j \omega_k - \omega(\nabla_j \partial_k))$$

$$= \partial_i \partial_j \omega_k - (\nabla_i \omega)(\cancel{\nabla_j \partial_k}) - \omega(\nabla_i \nabla_j \partial_k)$$

$$= \partial_i \partial_j \omega_k - \omega(\nabla_i \nabla_j \partial_k) \quad @ \; p$$

$$\Rightarrow (\nabla^2_{i,j} \omega - \nabla^2_{j,i} \omega)(\partial_k) = \cancel{\partial_i \partial_j \omega_k} - \omega(\nabla_i \nabla_j \partial_k) - \cancel{\partial_j \partial_i \omega_k} + \omega(\nabla_j \nabla_i \partial_k)$$

$$= -\omega(\nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k)$$

$$= -\omega(R(\partial_i, \partial_j) \partial_k) \quad @ \; p$$

since both sides tensorial

$$\Rightarrow (\nabla^2_{X,Y} \omega - \nabla^2_{Y,X} \omega)(Z) = X^i Y^j Z^k (\nabla^2_{i,j} \omega - \nabla^2_{j,i} \omega)(\partial_k)$$

$$= X^i Y^j Z^k (-\omega(R(\partial_i, \partial_j) \partial_k))$$

$$= -\omega(R(X, Y) Z)$$

Q4: choose normal coords x^i @ $0 \in M$

$$\Rightarrow \nabla \partial_i \big|_0 = 0$$

then: $\nabla_p R_{ijkl} = \partial_p R_{ijkl}$ @ 0

$$= \langle \nabla_p \nabla_i \nabla_j \partial_k - \nabla_p \nabla_j \nabla_i \partial_k , \partial_l \rangle \quad @ \ 0$$

$$\Rightarrow \nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl}$$

$$= \langle \nabla_p \nabla_i \nabla_j \partial_k - \nabla_p \nabla_j \nabla_i \partial_k + \nabla_i \nabla_j \nabla_p \partial_k - \nabla_i \nabla_p \nabla_j \partial_k + \nabla_j \nabla_p \nabla_i \partial_k - \nabla_j \nabla_i \nabla_p \partial_k , \partial_l \rangle$$

$$= R(\partial_p, \partial_i, \underbrace{\nabla_j \partial_k}_{0}, \partial_l) + R(\partial_j, \partial_p, \overbrace{\nabla_i \partial_k}^{0}, \partial_l) + R(\partial_i, \partial_j, \underbrace{\nabla_p \partial_k}_{0}, \partial_l)$$

$$= 0$$

by tensoriality, $(\nabla_T R)(X, Y, Z, W) + (\nabla_X R)(Y, T, Z, W) + (\nabla_Y R)(T, X, Z, W) = 0$ @ 0

Q5. let Q = quadratic bilinear form

$$\hookrightarrow \int_{S^{n-1}} Q(\theta, \theta)\, d\theta = \sum_{i,j} Q(e_i, e_j) \int_{S^{n-1}} (\theta \cdot e_i)(\theta \cdot e_j)\, d\theta$$

where $e_1 \cdots e_n$ = std basis of $\mathbb{R}^n$

if $i \neq j$, choose $R \in O(n)$ st. $R(e_i) = e_i$
$R(e_j) = -e_j$

$\hookrightarrow$ R isometry of S^{n-1}

and
$$\int_{S^{n-1}} (\theta \cdot e_i)(\theta \cdot e_j)\, d\theta = -\int_{R(S^{n-1})} (R(\theta) \cdot e_i)(R(\theta) \cdot e_j)\, d\theta$$

$$= -\int_{S^{n-1}} (\theta \cdot e_i)(\theta \cdot e_j)\, d\theta = 0$$

if $i = j$, choose $R \in O(n)$ st $Re_i = e_j$

$$\Rightarrow \int_{S^{n-1}} (\theta \cdot e_i)^2\, d\theta = \int_{R(S^{n-1})} (R(\theta) \cdot e_j)^2\, d\theta = \int_{S^{n-1}} (\theta \cdot e_j)^2\, d\theta$$

$$\Rightarrow n \int_{S^{n-1}} (\theta \cdot e_i)^2\, d\theta = \sum_{i=1}^{n} \int_{S^{n-1}} (\theta \cdot e_i)^2\, d\theta$$

$$= \int_{S^{n-1}} |\theta|^2\, d\theta = |S^{n-1}|$$

So $$\int_{S^{n-1}} Q(\theta,\theta)\, d\theta = \sum_{i,j} Q(e_i,e_j) \int_{S^{n-1}} (\theta\cdot e_i)(\theta\cdot e_j)\, d\theta$$

$$= \sum_i Q(e_i,e_i) \frac{|S^{n-1}|}{n}$$

$$= \frac{|S^{n-1}|}{n} \operatorname{tr} Q$$