Math 60670 Homework 6

Due Tuesday, March 5.

Problem 1: For a given linear connection ∇ on a Riemannian manifold (M, g), show that the following are equivalent:

A) ∇ is compatible with g;

B) $\nabla g \equiv 0$;

C) if V, W are vector fields along a curve $\gamma(t)$, then

$$\frac{d}{dt} g(V, W) = g(\frac{DV}{dt}, W) + g(V, \frac{DW}{dt});$$

D) if V, W are parellel vector fields along γ, then $g(V, W)$ is constant in t;

E) parallel translation is a linear isometry $T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$, for any times t_1, t_2.

Problem 2: Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be a diffeomorphism between connected Riemannian manifolds M, $\tilde{M}$.

A) Suppose ϕ is a Riemannian isometry. Show that $\phi \circ \exp_p = \exp_{\phi(p)} \circ D\phi|_p$ whereeever this is defined.

B) Let $\tilde{\phi}$ be another Riemannian isometry $(M, g) \to (\tilde{M}, \tilde{g})$, and suppose there is a point $p \in M$ so that $\phi(p) = \tilde{\phi}(p)$ and $D\phi|_p = D\tilde{\phi}|_p$. Show that $\phi = \tilde{\phi}$.

C) Show that ϕ is a Riemannian isometry if and only if ϕ preserves distances, i.e. $d_g(p, q) = d_{\tilde{g}}(\phi(p), \phi(q))$ for all $p, q \in M$.

D) (Bonus) Show that part C) holds even if ϕ is only assumed to be a distance-preserving homeomorphism.

Problem 3: A. Show that isometries between Riemannian manifolds take geodesics to geodesics.

B. Consider the half-space model of hyperbolic space $\mathbb{H}^2$, i.e. $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ equipped with the hyperbolic metric $g = (dx^2 + dy^2)/y^2$. Show that the geodesics of $(\mathbb{R}^2_+, g)$ are vertical half-lines and half-circles that intersect the "boundary" $\{y = 0\}$ orthogonally. Hint: Use part A and HW3 Problem 3.

C. Deduce that every geodesic in $\mathbb{H}^2$ can be extended for all time, i.e. $\mathbb{H}^2$ is "complete."