Q: let $X_1 = (1, 0)$

$X_2 = (1, x^2) \Rightarrow X_1 = X_2 = (1, 0)$ on x^1-axis

$Y = (0, 1)$

but $L_{X_1} Y = (0, 0)$ while $L_{X_2} Y = -\partial_2(x^2)\,\partial_2 = (0, -1)$

Q2. A. trivially T = $\mathbb{R}$-linear

need to show $T = C^\infty$-linear

$$T(fX,Y) = \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y]$$

$$= f\nabla_X Y - f\nabla_Y X - \cancel{Y(f)X} - f[X,Y] + \cancel{Y(f)X}$$

$$= f\,T(X,Y)$$

and $T(X,fY) = f\,T(X,Y)$ by anti-symmetry

B. let $(x^1, \dots, x^n)$ = std coords on $\mathbb{R}^n$

↳ $\frac{\partial}{\partial x^i} = e_i$ = constant

$$\Rightarrow T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = D_{\partial_i}\partial_j - D_{\partial_j}\partial_i - [\partial_i, \partial_j]$$

$$= D_{e_i}e_j - D_{e_j}e_i - \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right)$$

$$= 0 \qquad \text{since } e_i = \text{const}$$

$\Rightarrow T = 0$ by tensoriality

C. pick coords $(x^1, \dots, x^n)$

then $T = 0 \iff T(\partial_i, \partial_j) = 0 \quad \forall i,j$

$$\iff \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = 0 \quad \forall i,j$$

$$\iff (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k = 0 \quad \forall i,j$$

$$\iff \Gamma_{ij}^k = \Gamma_{ji}^k \quad \forall i,j,k$$

D. pick coords x^i

$$T=0 \Rightarrow \nabla^2_{ij} u = \partial_i \partial_j u - \Gamma_{ij}^{k} \partial_k u$$

$$= \partial_j \partial_i u - \Gamma_{ji}^{k} \partial_k u = \nabla^2_{ji} u$$

conversely, if $\nabla^2_{ij} u = \nabla^2_{ji} u$ then take $u = x^p$

$$\Rightarrow \nabla^2_{ij} x^p = \underbrace{\partial_i \cancel{\partial_j x^p}}_{0} - \Gamma_{ij}^{k} \partial_k x^p = -\Gamma_{ij}^{p}$$

$$\shortparallel$$

$$\nabla^2_{ji} x^p = \underbrace{\partial_j \cancel{\partial_i x^p}}_{0} - \Gamma_{ji}^{k} \partial_k x^p = -\Gamma_{ji}^{p}$$

$$\Rightarrow \Gamma_{ij}^{p} = \Gamma_{ji}^{p}$$

Q3: let us formally define $z = x + \sqrt{-1}\,y$, $\bar{z} = x - \sqrt{-1}\,y$

$$\longleftrightarrow \quad x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2\sqrt{-1}}$$

need to show $z \longmapsto A\cdot z$ = diffeo^n $\mathcal{U} \to \mathcal{U}$, preserving metric $\frac{dx^2+dy^2}{y^2}$

first check $A\cdot z$ maps $\mathcal{U}$ into $\mathcal{U}$:

$$\operatorname{Im}(A\cdot z) = \frac{A\cdot z - A\cdot\bar{z}}{2\sqrt{-1}} \quad \text{since } A \text{ real-valued}$$

$$= \left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)\frac{1}{2\sqrt{-1}}$$

$$= \frac{(ad-bc)\operatorname{Im}(z)}{(cz+d)(c\bar{z}+d)}$$

recall $\det A = 1$ since $A \in SL(2,\mathbb{R})$

$$= \frac{y}{(cx+d)^2+y^2} > 0 \text{ if } y > 0 \quad \checkmark$$

I claim $A\cdot z$ = group action ie $Id\cdot z = z$ and $A\cdot(A'\cdot z) = (AA')\cdot z$

check: $Id\cdot z = \frac{z}{1} = z$ ✓

$$A\cdot(A'\cdot z) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

$$= \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + db)} = (AA') \cdot z$$ ✓

so $A^{-1} \cdot z$ = inverse map to $A \cdot z \implies A \cdot z$ = diffeo $U \to U$

let us formally set $dz = dx + \sqrt{-1}\, dy$, $d\bar{z} = dx - \sqrt{-1}\, dy$

(being 1-forms on the complexified tangent bundle)

then if $f(x,y) \implies df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$

smooth function

where $f(x,y) = f(z, \bar z)$

via $x = \frac{z + \bar z}{2}$

$y = \frac{z - \bar z}{2\sqrt{-1}}$

and $\frac{dx^2 + dy^2}{y^2} = \frac{dz\, d\bar z}{\mathrm{Im}(z)}$

so $(A\cdot)^*\left(\frac{dx^2 + dy^2}{y^2}\right) = (A\cdot)^*\left(\frac{dz\, d\bar z}{\mathrm{Im}(z)^2}\right) = \frac{d(A\cdot z)\, d(\overline{A \cdot z})}{\mathrm{Im}(A\cdot z)^2}$

$$= \frac{d(A\cdot z)\, d(A \cdot \bar z)}{\mathrm{Im}(A\cdot z)^2}$$

$$d(A\cdot z) = \frac{\partial (A\cdot z)}{\partial z} dz + \cancel{\frac{\partial (A\cdot z)}{\partial \bar z} d\bar z}^{\to 0}$$

$$= \left(\frac{a}{cz+d} - \frac{(az+b)c}{(cz+d)^2}\right) dz = \frac{ad - bc}{(cz+d)^2} dz = \frac{dz}{(cz+d)^2}$$

and similarly $d(\overline{A\cdot z}) = d(A\cdot \bar{z}) = \dfrac{d\bar{z}}{(c\bar{z}+d)^2}$

$$\Rightarrow (A\cdot)^*\left(\frac{dx^2+dy^2}{y^2}\right) = \frac{d(A\cdot z)\,d(A\cdot \bar{z})}{\mathrm{Im}(A\cdot z)^2}$$

$$= \frac{dz\,d\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}\;\frac{1}{\dfrac{\mathrm{Im}(z)^2}{(cz+d)^2(c\bar{z}+d)^2}}$$

$$= \frac{dz\,d\bar{z}}{\mathrm{Im}(z)^2} = \frac{dx^2+dy^2}{y^2}$$

So $A\cdot z$ = preserves metric $\longrightarrow$ isometry