Math 60670 Final

Due by 11:59pm Thursday May 9th. Send your exam to `nedelen@nd.edu`. You may use Lee's and do Carmo's books, the class notes, and previous homeworks and midterms from this class, but no other resources. You are allowed to quote theorems/lemmas/corollaries from the book or class or previous homework *that we or the book have proven.* You are not allowed to quote statements from the books that are given without proof (e.g. exercises).

Q1: A. Let (M^n, g) be a complete Riemannian manifold ($n \geq 2$), and suppose M admits a geodesic line, i.e. there is a geodesic $\gamma : \mathbb{R} \to M$ parameterized by arclength which is minimizing on any finite segment. By considering variations of the form $\phi(t)e_i(t)$, where $\{e_i(t)\}_i$ is an ON parallel frame along γ, show that the following inequality must be true:

$$\int_{\mathbb{R}} (n-1)\phi'(t)^2 - \mathrm{Ric}|_{\gamma(t)}(\gamma', \gamma')\phi(t)^2 dt \geq 0 \quad \forall \phi \in C^1_c(\mathbb{R}).$$

B. Using part A, deduce that M cannot have positive Ricci curvature.

C. Use parts A, B to prove that any complete, non-compact (M^n, g) with positive Ricci curvature can have only one end. (Hint: use a result from the Midterm).

C. Prove that that $S^1 \times \mathbb{R}$ admits no complete metric of positive sectional curvature.

D. On the other hand, show that $\mathbb{R}^2$ does admit a complete metric of positive sectional curvature. (Hint: use a result from Homework 9).

Q2: Let (M^n, g) be a complete, connected Riemannian manifold, and $f : M \to \mathbb{R}$ a smooth function. Write ∇f for the gradient of f.

A. Suppose that $\nabla f \neq 0$ for all $x \in f^{-1}(0)$. Show that $S := f^{-1}(0)$ is a smooth embedded hypersurface, $\frac{\nabla f}{|\nabla f|}$ is a choice of unit normal for S, and the second fundamental form of S (with the induced metric) can be expressed as

$$B(X, Y) = -\frac{\nabla^2 f(X, Y)}{|\nabla f|^2} \nabla f.$$

B. Suppose that $|\nabla f| \equiv 1$. Prove that the integral curves of f are geodesics. Deduce that the flow $\phi_t(x)$ of ∇f exists for all $t \in \mathbb{R}$. Bonus: show the integral curves are minimizing geodesics.

C. Suppose that $\nabla^2 f \equiv 0$, and f is not constant. Prove that M isometrically splits off a line, in the sense that there is an $(\hat{M}^{n-1}, \hat{g})$ so that (M, g) is isometric to $(\hat{M} \times \mathbb{R}, \hat{g} + dr)$. Hint: Use a result from the Midterm.

Q3: Let (M^2, g) be a 2-dimensional complete, simply-connected Riemannian manifold with sectional curvature $K \leq 0$. Fix $p \in M$, and let $A(r) = \text{area}_M(B_r(p))$, $L(r) = \text{length}_M(\partial B_r(p))$.

A. Show that $A(r)$ is a smooth function of $r \in (0, \infty)$, and $A'(r) = L(r)$, $L'(r) = \int_{\partial B_r(p)} k ds$, where k is the geodesic curvature of $\partial B_r(p)$ w.r.t. the inwards normal.

B. Prove the Euclidean isoperimetric inequality: $4\pi A(r) \leq L(r)^2$. Hint: Consider the function $f(r) = L(r)^2 - 4\pi A(r)$.

C. Show that if at some radius equality holds $4\pi A(r) = L(r)^2$, then $B_r(p)$ is isometric to the flat Euclidean ball of radius r in $\mathbb{R}^2$.

Q4: Is it possible (prove or disprove) to find a complete *non-planar* surface $S \subset \mathbb{R}^3$ so that:

A. For every $p \in S$, there is a line in $\mathbb{R}^3$ passing through p contained in S?

B. For every $p \in S$, there are two lines in $\mathbb{R}^3$ passing through p contained in S?

C. For every $p \in S$, there are three lines in $\mathbb{R}^3$ passing through p contained in S?