Differential Geometry

refresher: V = n-dim vector space

V^* = dual (= real-valued linear functions $V \to \mathbb{R}$)

covariant k-tensor on V = multilinear map $\underbrace{V \otimes V \cdots \otimes V}_{k\text{-copies}} \to \mathbb{R}$

contravariant ℓ-tensor = multilinear map $\underbrace{V^* \otimes \cdots \otimes V^*}_{\ell\text{-copies}} \to \mathbb{R}$

note: $V \cong V^*$ but not "naturally"

↳ E_i basis for $V \rightsquigarrow$ dual basis ω^i for V^*

where $\omega^i(E_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

note: natural pairing $V \otimes V^* \to \mathbb{R}$

$(v, \omega) \mapsto \omega(v)$

note: V = "column vectors", V^* = "row vectors"

define (k,ℓ)-tensor = multi-linear map $V^{\otimes k} \otimes (V^*)^{\otimes \ell} \to \mathbb{R}$

↳ $T^{(k,\ell)}(V)$ = space of (k,ℓ)-tensors

any (k,ℓ)-tensor $F = \sum F^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \underline{E_{j_1} \otimes \cdots \otimes E_{j_\ell}} \otimes \underline{\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}}$

in basis E_i, ω^i

covariant = "change with basis" = lower indices contravariant = "change against basis" = upper indices

e.g. if $\underline{\tilde{E}_i = A_i^{\,j} E_j}$.

$$\text{"} \underbrace{\omega^i}_{\tilde{\omega}^i} \underbrace{A^{-1} A E_j}_{\tilde{E}_j} = \delta^i_j \text{"}$$

or $\underline{\tilde{\omega}^i = (A^{-1})^i_j \, \omega^j}$

(Einstein summation = sum over repeated indices)

if $X = X^i E_i$ vector

$$= \underbrace{X^i (A^{-1})_i^{\,j}}_{\tilde{X}^j} \tilde{E}_j$$

alternating tensor = anti-symmetric

i.e. $F_{i_1 i_2 \dots} = -F_{i_2 i_1 \dots}$

$F^{j_1 j_2 \dots} = -F^{j_2 j_1 \dots}$

$\Lambda^k(V)$ = alternating covariant k-tensor (maps $V^{\otimes k} \to \mathbb{R}$)

= k-forms

$\dim \Lambda^k(V) = \binom{n}{k}$

$\dim T^{(k,\ell)}(V) = (k+\ell)^n$

wedge product: $\Lambda^k(V) \otimes \Lambda^\ell(V) \longrightarrow \Lambda^{k+\ell}(V)$

$(\omega, \tau) \longmapsto \omega \wedge \tau$

$$(\omega \wedge \tau)(v_1 \cdots v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \text{ perm}} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \tau(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma = \text{even \# of transpositions} \\ -1 & \text{if } \sigma = \text{odd \# } \ldots \end{cases}$$

$$(\omega \wedge \tau)_{i_1 \cdots i_{k+l}} = \underbrace{\varepsilon^{j_1 \cdots j_{k+l}}_{i_1 \cdots i_{k+l}}}_{\text{"Levi-Civita symbol"}} \omega_{j_1 \cdots j_k} \tau_{j_{k+1} \cdots j_{k+l}} \frac{1}{k!\,l!}$$

Ex: if ω^i = dual basis to the std basis in $\mathbb{R}^n$

$= (0, \ldots, 1, \ldots 0)$
$\quad\quad$ i^{th} slot

$\omega^1 \wedge \cdots \wedge \omega^n = \det$

note: not every collection of numbers $a_{ijk\ldots}$ is a tensor

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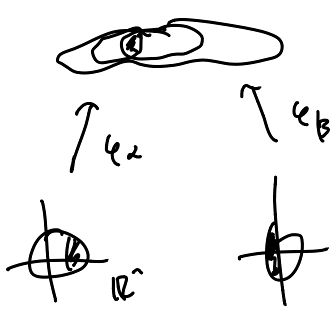
smooth n-dim manifold M = set M with a collection of injective maps $\{\varphi_\alpha : U_\alpha \to M\}_\alpha$, $U_\alpha \subset \mathbb{R}^n$ open

s.t. ① $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$



② if $W = \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$

then $\varphi_\alpha^{-1}(W), \varphi_\beta^{-1}(W)$ open in $\mathbb{R}^n$

and $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(W) \to \varphi_\beta^{-1}(W)$ = smooth diffeomorphism

(③ $\{\varphi_\alpha, U_\alpha\}$ is "maximal")

$\hookrightarrow \varphi_\alpha, U_\alpha$ = coord chart $\qquad$ identify $U_\alpha \longleftrightarrow \varphi_\alpha(U_\alpha)$

$\{(\varphi_\alpha, U_\alpha)\}_\alpha$: atlas

$(x^1, \dots, x^n)$ coords in $U_\alpha \longleftrightarrow$ think of pts $\varphi_\alpha(x^1, \dots, x^n)$

(forget φ_α)

recall: tangent space T_xM of M @ $x \in M$

(= space of derivations of functions near x)

(= space of velocity vectors for curves passing thru x)

given coords $(x^1, \dots, x^n)$ near x

$\hookrightarrow$ the differentials $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ span T_xM

ie, if $f : M \to \mathbb{R}$ smooth

and $\alpha(t) = (x^1(t), \dots, x^n(t))$ some curve with $\alpha(0) = x$

the $\frac{d}{dt}\Big|_{t=0} (f \circ \alpha)(t) = \dot{x}^i \frac{\partial f}{\partial x^i}$

$\hookrightarrow T_xM$ = span of $\left(\frac{\partial}{\partial x^i}\right)_i = (\partial_i)_i$

T^*_xM = cotangent space (= dual to T_xM)

= space of covectors

= span of $(dx^1, \dots, dx^n)$ where $dx^i(\partial_j) = \delta_{ij}$

vector bundles $\qquad$ (lecture 2)

M = smooth n-mfld

E = smooth k-dim vector bundle over M

$$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \iff E \text{ smooth } (n+k)\text{-dim mfld}$$

surjective smooth $\pi : E \to M$

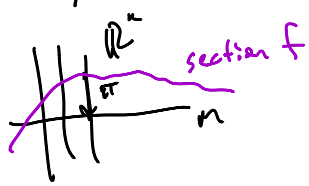
① $\pi^{-1}(p) = E_p$ = k-dim vector space (fiber)

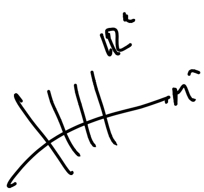
locally $M \times \mathbb{R}^k$

② @ every $p \in M$, $\exists$ chart $\varphi : U \times \mathbb{R}^k \to E$

s.t. $\pi(\varphi(x,v)) = \varphi(x,0)$

and $\varphi\big|_{\{q\}\times\mathbb{R}^k}$ = linear iso^n $\mathbb{R}^k \to E_{\varphi(q)}$





section of E = smooth function $f : M \to E$ s.t. $\pi \circ f = id$

$\Gamma(E)$ = space of sections

$TM = \bigcup_x T_x M$ is n-dim vector bundle over M

if (φ, U) chart for M, giving coords (x^i)

$(\Phi, U \times \mathbb{R}^n)$ chart for TM

$$(x, v) \longmapsto \left(\varphi(x), v^i \frac{\partial}{\partial x^i}\Big|_x\right)$$

$T^*M = \bigcup_x T^*_x M$ bundle

vector field field on M = section of TM i.e. $p \mapsto X(p) \in T_pM$

$\left(= \mathfrak{X}(M) \right)$

covector field field = 1 form = section $T^*M = \mathfrak{X}^*(M)$

$T^{(p,q)}(M) = \bigcup_{x \in M} T^{(p,q)}(T_xM)$ = tensor bundle

locally, in coordinates, looks like $T^{j_1 \dots j_k}_{i_1 \dots i_p} \partial_{j_1} \otimes \dots \otimes \partial_{j_k} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$

$\Lambda^k(M) = \bigcup_{x \in M} \Lambda^k(T_xM)$ = k-form on M

in coords: $\omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

(k,ℓ)-tensor field on M = section of $T^{(k,\ell)}(M)$

T "

in coords: $T(x)^{j_1 \dots j_\ell}_{i_1 \dots i_k} \partial_{j_1} \otimes \dots \otimes \partial_{j_\ell} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_k}$

fact: T is a tensor field $\in \Gamma(T^{(k,\ell)}(M))$

$\Leftrightarrow$ T is $\mathbb{R}$-multilinear map from k vector fields and ℓ 1-forms

which is $C^\infty(M)$-linear

$T(X_1, \dots, X_k, \omega_1, \dots, \omega_\ell)$ multilinear over $\mathbb{R}$

but $T(f(x)X_1, \dots) = f(x)T(X_1, \dots)$

__Lie Bracket:__ $X, Y \in \mathfrak{X}(M)$ vector fields

$\hookrightarrow [X,Y] = XY - YX \in \mathfrak{X}(M)$

i.e. $[X,Y]f = X(Y(f)) - Y(X(f))$

in coords $X = X^i \partial_i$ then $[X,Y] = X^i \partial_i (Y^j \partial_j) - Y^i \partial_i (X^j \partial_j)$

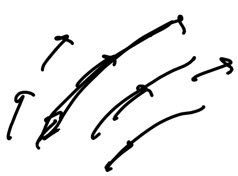
$Y = Y^j \partial_j$

$$= (X^i \partial_i Y^j - Y^i \partial_i X^j)\, \partial_j$$

$$\cancel{+ X^i Y^j \partial_i \partial_j - X^j Y^i \partial_i \partial_j} = 0$$

__Lie derivative:__ ODE $\Rightarrow$ solve $\begin{cases} \partial_t \varphi_t(p) = X(\varphi_t(p)) \\ \varphi_0(p) = p \end{cases}$

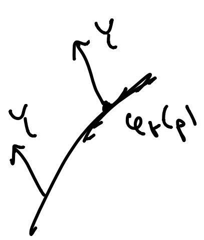
for $t \in (-\varepsilon, \varepsilon)$



smooth in t, p

Lie derivative of Y w.r.t. X

$$= L_X Y\big|_p = \lim_{t \to 0} \frac{D\varphi_{-t}\, Y(\varphi_t(p)) - Y(p)}{t}$$



__fact:__ $L_X Y = [X,Y]$

__thm__ (Frobenius): there a smooth function $\varphi_{t,s}(p)$

s.t. $\partial_t \varphi = X$, $\partial_s \varphi = Y$

$\Leftrightarrow [X,Y] = 0$

note: $[X,Y]$ not tensorial since $[X,fY]$

$$= X(fY) - fY(X)$$
$$= f[X,Y] + X(f)Y$$

Riemannian metric g on M (= smooth n-mfld)

= smooth choice of inner product on M

= section of $T^{(2,0)}(M)$ i.e. eats 2 vector fields

s.t. $g|_x$ = inner product on T_xM

i.e. if $X, Y \in \mathfrak{X}(M)$, $g(X,Y)$ smooth

in coords: $g = g_{ij}(x)\, dx^i dx^j$

where $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$

where g_{ij} = symmetric, positive definite for each x

notation: $ds^2 = g_{ij}\, dx^i dx^j$ (ds = inf. length element)

Ex: $(\mathbb{R}^n, g_{eucl})$, $g_{eucl} = (dx^1)^2 + \dots + (dx^n)^2$

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Ex: $(S^n, g_{round}) \subset (\mathbb{R}^{n+1}, g_{eucl})$

$$g_{round} = g_{eucl}\big|_{TS^n}$$

↳ more generally, if M is embedded in (N, g_N)

can restrict ambient metric g_N to M

→ get metric on M

Ex: T^n = n torus = $\mathbb{R}^n / \mathbb{Z}^n$ = $\mathbb{R}^n / (x^1, \dots, x^n) \sim (x^1 + n_1, x^2 + n_2, \dots)$ $n_1, n_2, \dots \in \mathbb{Z}$

$= (S^1)^n$

flat metric = metric inherited from quotient

↳ in general, if G = discrete group of isometries of (N, g)

↳ N/G (if mfd) inherits metric of N

Ex: $\varphi: M \longrightarrow (N, g)$ immersion

↳ $\varphi^* g$ = pullback metric on M

$(\varphi^* g)(X, Y) = g(D\varphi X, D\varphi Y)$

Ex product $(M_1, g_1) \times (M_2, g_2) = (M_1 \times M_2, g_1 + g_2)$

if (x^i) coords of M_1

(y^α) coords of M_2

$g_{M_1 \times M_2} = g_{ij} dx^i dx^j + g_{\alpha\beta} dy^\alpha dy^\beta$

ie. $\begin{vmatrix} g_1 & 0 \\ 0 & g_2 \end{vmatrix}$

if $\varphi : (M, g_M) \to (N, g_N)$ diffeomorphism

φ = <u>isometry</u> if $\varphi^* g_N = g_M$

$x^i \rightsquigarrow y^\alpha$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}$$

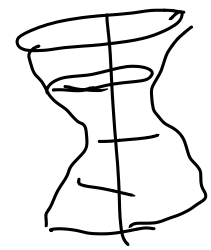
(lecture 3)

<u>Ex</u>: warped product

I = open interval, $f : I \to \mathbb{R}$ positive, smooth

(M^{n-1}, g_M) Riemannian mfld

$I \times_f M = (I \times M, dr^2 + f(r)^2 g_M)$



eg ① $((0,\infty) \times S^{n-1}, dr^2 + r^2 g_{S^{n-1}})$

$\Downarrow$ = polar coords on $(\mathbb{R}^n \setminus \{0\}, g_{eucl})$

$(r, \theta) \mapsto r\theta \in \mathbb{R}^n$

② $((0,\pi) \times S^{n-1}, dr^2 + \sin^2 r \, g_{S^{n-1}}) \cong (S^n \setminus \{\text{poles}\}, g_{round})$

③ $((0,\pi) \times S^{n-1}, dr^2 + \sinh^2 r \, g_{S^{n-1}}) \cong (\mathbb{H}^n \setminus \{pt\}, g_{hyperbolic})$

<u>basic Q</u>: how to tell if $(M_1, g_1) \cong (M_2, g_2)$?

<u>thm</u>: every smooth manifold admits a Riemannian metric

proof: take $\{\varphi_\alpha, U_\alpha\}$ atlas s.t. $\{\varphi_\alpha(U_\alpha)\}_\alpha$ loc. finite intersection

↳ choose partition of unity subordinate to $\varphi_\alpha(U_\alpha)$

$\{f_\alpha\}$

i.e. $\mathrm{spt}\, f_\alpha \subset \varphi_\alpha(U_\alpha)$, $0 \le f_\alpha \le 1$, $\sum_\alpha f_\alpha = 1$

$\varphi_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ ⇝ let $g_\alpha = (\varphi_\alpha^{-1})^* g_{eucl}$ on $\varphi_\alpha(U_\alpha)$

⇒ define $g = \sum_\alpha f_\alpha g_\alpha$ □

basic properties / consequences of metric g

- g gives natural iso^n $TM \leftrightarrow T^*M$

↳ $X \in T_xM$ ⇝ $X^\flat := g(X, \cdot) \in T_x^*M$

$\omega \in T_x^*M$ ⇝ $\omega^\# :=$ vector s.t. $g(\omega^\#, Y) = \omega(Y)$

i.e. $(\omega^\#)^\flat = \omega$

in coords: $(x^1, \dots, x^n)$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

define $g^{ij} =$ inverse of g_{ij}

if $X = X^i \frac{\partial}{\partial x^i}$ @ x then by definition $X^\flat = (X^\flat)_i\, dx^i$

$Y = Y^j \frac{\partial}{\partial x^j}$

$(X^\flat)_i Y^i = g_{ij} X^j Y^i$ for all Y

$\Rightarrow (X^\flat)_i = g_{ij} X^j$

"lowering index of X"

given $\omega = \omega_i\, dx^i$ then $g(\omega^\#, Y) = \omega(Y)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel \quad\quad\quad\quad\quad \shortparallel$

$$g_{ij}(\omega^\#)^j Y^i \qquad \omega_i Y^i$$

so $\omega_i = g_{ij}(\omega^\#)^j$

multiply by g^{pi} $\Rightarrow$ $g^{pi}\omega_i = \underline{g^{pi} g_{ij}}(\omega^\#)^j$

$$= \delta_{pj}(\omega^\#)^j$$

$$= (\omega^\#)^p$$

so: $\underline{(\omega^\#)^i = g^{ij}\omega_j}$ "raising index of ω"

- gradient vector field for $f \in C^\infty(M)$

$= \operatorname{grad} f := df^\#$

i.e. $g(\operatorname{grad} f, Y) = Y(f)$ $\big(= df(Y)\big)$

in coords: $(\operatorname{grad} f)^i = g^{ij}\dfrac{\partial f}{\partial x^j}$ $\qquad df = \dfrac{\partial f}{\partial x^i}dx^i$

e.g. in polar coords $(0,\infty) \times S^{n-1}$, $dr^2 + r^2 g_{S^{n-1}}$

$r \qquad \theta^1, \dots, \theta^{n-1}$

$$\Rightarrow g = dr^2 + r^2 d\theta^{1\,2} + \dots + r^2(d\theta^{n-1})^2 \quad @\,p$$

$$\begin{bmatrix} 1 & \\ & r^2 g_S \end{bmatrix}$$

$\dfrac{\partial}{\partial \theta^i}$ ON in S^{n-1} @ p

$$g^{-1} = dr^2 + r^{-2} d\theta^{1\,2} + \dots + r^{-2}(d\theta^{n-1})^2 \quad @\,p$$

$$\Rightarrow \text{gradient of } f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta^i}\frac{\partial}{\partial \theta^i} \quad @\,p$$

- raise / lower indices of a tensor

e.g. $T_{ijk} \rightsquigarrow T_i{}^j{}_k = g^{jp} T_{ipk}$

↳ can take trace against metric

e.g. if h = symmetric $(2,0)$-tensor h_{ij}

$$\operatorname{tr}_g h = h^i{}_i = g^{ij} h_{ij}$$

$$= \sum h(E_i, E_i) \quad \text{for } E_i \text{ ON basis}$$

- g gives inner product on tensor

spse T, S are (k, l)-tensors on $T_x M$

E_i = g-ON basis for $T_x M$

θ^i = dual basis (so $\theta^i(E_j) = \delta_{ij}$)

$$g(T,S) = \sum T(E_{i_1}, \dots, E_{i_k}, \theta^{j_1}, \dots \theta^{j_l}) \, S(E_{i_1}, \dots, E_{i_k}, \theta^{j_1} \dots \theta^{j_l})$$

if $T = T_{i_1 \dots i_k}^{j_1 \dots j_l} \, dx^{i_1} \otimes \dots \partial_{j_1} \otimes \dots \qquad S = S_{i_1 \dots i_k}^{j_1 \dots j_l}$

$$g(T,S) = g^{i_1 r_1} \dots g^{i_k r_k} g_{j_1 t_1} \cdots g_{j_l t_l} T_{i_1 \dots i_k}^{j_1 \dots j_l} S_{r_1 \dots r_k}^{t_1 \dots t_l}$$

($\Rightarrow$ dual basis is ON in $T^*_x M$)

- g gives metric space structure on (M, g)

if $\gamma : [a,b] \longrightarrow M$ (piecewise) smooth curve

↳ $\operatorname{length}(\gamma) = \int_a^b g(\gamma', \gamma')^{1/2} \, dt$

can define $d_g(p,q) = \inf \left\{ \text{length}(\gamma) : \gamma = \text{p.w. smooth curve } \gamma:[0,1] \to M, \ \gamma(0)=p, \ \gamma(1)=q \right\}$

$\to (M, d_g) =$ metric space

- g gives notion of volume on M^n

if $M^n =$ oriented, $g =$ Riemannian metric

$\Rightarrow \exists!$ "volume form" $dV = n$-form on M

st. $dV(E_1, E_2, \dots E_n) = 1$

for $E_i =$ oriented, ON basis

exercise: in coords, $dV = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n$

$\Rightarrow$ can integrate functions $\int f \, dV = \int f \sqrt{\det g}\, dx^1 \wedge \dots \wedge dx^n$

(lecture 4)

(M, g) n-dim Riemannian mfd

notation: $g(X,Y) = \langle X, Y \rangle$, $|X| = \langle X, X \rangle^{1/2}$

lemma: if $F_1 \dots F_n =$ smooth vector fields in $U \subset M$

lin ind @ each $p \in U$

$\Rightarrow \exists E_1 \dots E_n =$ smooth vector fields in U

g-ON @ each $p \in U$

proof: do Gram-Schmidt $E_1 = \dfrac{F_1}{|F_1|}$ ← smooth positive function

$$E_2 = \frac{F_2 - \langle F_2, E_1\rangle E_1}{|F_2 - \langle F_2, E_1\rangle E_1|}$$

$\square$

....

<u>lemma:</u> given pt $o \in M$, $\exists$ coords (y^i) s.t. $\left.\frac{\partial}{\partial y^i}\right|_o$ = g-orth @ o

<u>proof:</u> start with coords (x^i) near o (wlog $x^i(o) = 0$)

define (y^i) by $x^i = a^i_j y^j$ for a^i_j = <u>const</u> matrix (invertible)

$$\hookrightarrow \frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} = a^i_j \frac{\partial}{\partial x^i}$$

$$\Rightarrow g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = g\left(a^p_j \frac{\partial}{\partial x^p}, a^q_k \frac{\partial}{\partial x^q}\right)$$

$$= a^p_j a^q_k \left. g_{pq}\right|_o \underset{\text{want}}{=} \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & \text{else}\end{cases}$$

think about a^i_j = entries of matrix $A = (A_{ij})$

$$(AB)_{ij} = A_{ik} B_{kj}$$

$$(A^T g|_o A)_{jk} \underset{\text{want}}{=} \delta_{jk} = I_{jk}$$

g inner product $\Rightarrow$ can write $g|_o = P^T \Sigma P = P^T \sqrt{\Sigma}\sqrt{\Sigma} P$

for P = orthogonal matrix

Σ = diagonal matrix, +ve entries

so let $A = P^T(\sqrt{\Sigma}^{-1})$

$$\Rightarrow A^T g|_0 A = (\sqrt{\Sigma})^{-1} P \underbrace{P^T \sqrt{\Sigma}\sqrt{\Sigma} P}_{g|_0} P^T (\sqrt{\Sigma})^{-1}$$

$$= I \quad \checkmark \qquad \square$$

note: $\frac{\partial}{\partial y^i}$ g-ON in neighborhood U $\Leftrightarrow$ (U, g) isometric to flat space

Connections

how to differentiate vector fields? (vectors @ diff. pts live in diff. tangent spaces)

in $\mathbb{R}^n$: $Y(x^1, \dots, x^n) = (Y^1(x), \dots, Y^n(x))$ smooth vector field in $\mathbb{R}^n$

$$X = (X^1, \dots, X^n)$$

$\Rightarrow$ directional derivative $D_X Y\big|_p = \frac{d}{dt}\Big|_{t=0} Y(p + tX)$

$$(D_X Y)^i = \frac{\partial Y^i}{\partial x^j}\Big|_p X^j$$

note: satisfies Leibnitz rule in Y, tensorial in X, linear (over $\mathbb{R}$) in Y

$D_X(fY) = X(f)Y + f D_X Y$

f smooth function

<u>already seen</u>: • $\mathcal{L}_X Y = [X,Y]$ not tensorial in X (or Y)

very rigid, ind. of metric

[if $A = (2,1)$-tensor, could define $\tilde{D}_X Y = D_X Y + A(X,Y)$ ...]

• d for ω = 1-form doesn't spit out vector... very rigid...

<u>Defn:</u> a <u>connection</u> ∇ on a vector bundle $E \longrightarrow M$

= smooth map $\nabla : \underline{\mathfrak{X}(M) \times \Gamma(E)} \longrightarrow \Gamma(E)$

① (tensorial in X): $\nabla_{f_1 X_1 + f_2 X_2} \sigma = f_1 \nabla_{X_1} \sigma + f_2 \nabla_{X_2} \sigma$

$f_1, f_2 \in C^\infty(M)$, $X_1, X_2 \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$

② (Leibniz rule): $\nabla_X (f\sigma) = X(f)\,\sigma + f\,\nabla_X \sigma$

③ (linear in σ /$\mathbb{R}$): $\nabla_X (a_1 \sigma_1 + a_2 \sigma_2) = a_1 \nabla_X \sigma_1 + a_2 \nabla_X \sigma_2$

$a_1, a_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \Gamma(E)$

<u>linear connection ∇ on M</u> = <u>connection on TM</u>

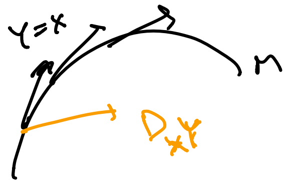
i.e. $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

<u>note</u>: not a tensor!!

Ex: $D_X Y$, $D^A_X Y$ both connections on $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

Ex: M^n = submanifold of $\mathbb{R}^{n+k}$

Can define $\nabla_X Y\big|_p = \mathrm{proj}_{T_pM}\left(D_X Y\big|_p\right)$



(more generally can restrict connection $\overline{\nabla}$ of N to connection in $M \subset N$)

Rmk: connection ∇ is a local operator

if $\sigma_1, \sigma_2 \in \Gamma(E)$ s.t. $\sigma_1 \equiv \sigma_2$ near $p \in M$

take $\varphi \in C^\infty_c(M)$ s.t. $\varphi \equiv 1$ near p, $\varphi = 0$ outside of $\{\sigma_1 = \sigma_2\}$

then $\nabla_X(\varphi\sigma_1) = X(\varphi)\sigma_1 + \varphi\nabla_X\sigma_1$

$\qquad @p = 0 + \nabla_X\sigma_1|_p$

$\qquad ||$

$\nabla_X(\varphi\sigma_2) = X(\varphi)\sigma_2 + \varphi\nabla_X\sigma_2$

$\qquad @p = 0 + \nabla_X\sigma_2|_p$

(and tensorial in $X \Rightarrow$ depends only on $X|_p$)

linear connections on M (= smooth manifold) (lecture 5)

M, linear connection ∇ (= connection on TM)

$(x^1, \dots, x^n)$ coords on M, define $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$

Christoffel symbols
(smooth functions, <u>not</u> tensors)

$\hookrightarrow$ $X = X^i \partial_i$, $Y = Y^j \partial_j$

then $\nabla_X Y = X^i \nabla_{\partial_i}(Y^j \partial_j)$

$= X^i(\partial_i Y^j)\,\partial_j + X^i Y^j \Gamma_{ij}^k \partial_k$

(twisted version of Euclidean $D_X Y$)

<u>lemma:</u> linear connection ∇ induces "compatible" connection on $T^{(k,\ell)}(M)$ by product rule

satisfies: ① agrees with ∇ on $T^{(0,1)}(M)$

② $\nabla(T \otimes S) = (\nabla T)\otimes S + T \otimes (\nabla S)$

③ $\nabla \operatorname{tr} T = \operatorname{tr} \nabla T$

<u>proof:</u> given $\omega \in \mathfrak{X}^*(M)$

define $\nabla\omega$ by: $\underset{\text{fnctn}}{X(\omega(Z))} = (\nabla_X \omega)(Z) + \omega(\nabla_X Z)$

$(X, Z \in \mathfrak{X}(M))$ so $\underline{\nabla_X \omega} \in \mathfrak{X}^*(M)$

i.e. $(\nabla_X\omega)(Z) = X(\omega(Z)) - \omega(\nabla_X Z)$

given $T \in \Gamma(T^{(k,\ell)}(M))$

$\Rightarrow$ define $X(T(Z_1,\dots,Z_k,\omega_1,\dots,\omega_\ell))$

$= (\nabla_X T)(Z_1,\dots,Z_k,\omega_1,\dots,\omega_\ell)$

$+ T(\nabla_X Z_1, Z_2, \dots) + T(Z_1, \nabla_X Z_2, Z_3, \dots)$

$+ T(Z_1,\dots,Z_k, \nabla_X \omega_1, \omega_2, \dots)$

in coords: $\underline{\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k}$ $\qquad dx^i(\partial_j) = \delta_{ij}$

$$(\nabla_{\partial_i} \underline{dx^j})(\partial_k) = \partial_i\left(dx^j(\partial_k)\right) - dx^j(\nabla_{\partial_i} \partial_k)$$
$$= \underbrace{\partial_i(\delta_{jk})}_{0} - \Gamma^p_{ik} \underbrace{dx^j(\partial_p)}_{\delta_{jp}}$$
$$= -\Gamma^j_{ik}$$

i.e. $\underline{\nabla_{\partial_i} dx^j = -\Gamma^j_{ik} dx^k}$

eg.

$Y^i_j = (1,1)$-tensor i.e. $Y(X, \omega)$

product rule: $\partial_p Y^i_j = \partial_p\left(Y(\partial_j, dx^i)\right)$
$$= (\nabla_p Y)(\partial_j, dx^i) + Y(\nabla_p \partial_j, dx^i) + Y(\partial_j, \nabla_p dx^i)$$
$$= (\nabla_p Y)(\partial_j, dx^i) + \Gamma^k_{pj} Y(\partial_k, dx^i) - \Gamma^i_{pk} Y(\partial_j, dx^k)$$
$$= \underline{\nabla_p Y^i_j + \Gamma^k_{pj} Y^i_k - \Gamma^i_{pk} Y^k_j}$$

$\mathrm{tr}\, Y = Y^i_i \quad \left(= \sum_i Y(\partial_i, dx^i)\right)$

$$\nabla_p \mathrm{tr}\, Y = \partial_p Y^i_i = \nabla_p Y^i_i + \underline{\Gamma^k_{pi} Y^i_k} - \underline{\Gamma^i_{pk} Y^k_i}$$
$$= \nabla_p Y^i_i + \cancel{\Gamma^k_{pi} Y^i_k - \Gamma^k_{pi} Y^i_k} \to 0$$
$$= \mathrm{tr}(\nabla_p Y)$$

$\square$

note: $T \in \Gamma T^{(k,l)}(M)$

$\nabla_\bullet T \in \Gamma T^{(k+1,l)}(M)$

e.g. $u \in C^\infty(M) \longrightarrow \underset{=(1,0)\text{ tensor}}{\nabla u}$ takes $X \mapsto \nabla_X u \equiv X(u)$

$\longrightarrow \nabla^2 u = (2,0)$ tensor $=$ Hessian of u

$\longrightarrow \nabla^2 u(X,Y) = X(\nabla_Y u) - \nabla_{\nabla_X Y} u$

$= \underline{X(Y(u)) - (\nabla_X Y)(u)}$

<u>tensors associated</u> with a linear connection:

① difference $\nabla - \hat{\nabla} = (2,1)$ - tensor

② torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = (2,1)$ - tensor

③ curvature <u>tensor</u> $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$= (3,1)$ - tensor

check <u>first</u>: $(X, Y, \omega) \mapsto \underline{\omega}(\nabla_X Y - \hat{\nabla}_X Y)$

linear over $\mathbb{R}$

$\rightarrow$ check linear over $C^\infty(M)$

$(X, fY, \omega) = \omega(\nabla_X(fY) - \hat{\nabla}_X(fY))$

$= \omega(f\nabla_X Y + \cancel{X(f)Y} - f\hat{\nabla}_X Y - \cancel{X(f)Y})$

$= f\,\omega(\nabla_X Y - \hat{\nabla}_X Y)$ ✓

what does torsion mean?

lemma: $x \in M$, ∇ linear connection

$$\Rightarrow \exists\, F_1 \dots F_n \in \mathfrak{X}(M) \text{ s.t. } \nabla F_i\Big|_x = 0$$

$F_i\Big|_x$ give basis for T_xM

proof: take (x^i) coords near $x = 0$

$\hookrightarrow \frac{\partial}{\partial x^i} = \partial_i$ = coord. basis

let $F_i^j = a_i^j(x)\,\partial_j$ for $a_i^j(x) = \underline{\delta_{ij} + a_{ik}^j}\, x^k$ consts.

$$\nabla_{\partial_p} F_i\Big|_0 \overset{\text{want}}{=} 0$$

$$= \nabla_{\partial_p}(a_i^j\,\partial_j)$$

$$= \partial_p a_i^j\,\partial_j + a_i^j\,\Gamma_{pj}^k\,\partial_k$$

$$@\,0 \quad = \partial_p(\delta_{ij} + a_{ik}^j x^k)\Big|_0 \partial_j + \delta_{ij}\,\Gamma_{pj}^k\Big|_0 \partial_k$$

$$= a_{ik}^j\,\delta_{kp}\,\partial_j + \Gamma_{pi}^k\Big|_0 \partial_k$$

$$= \left(a_{ip}^k + \Gamma_{pi}^k\Big|_0\right)\partial_k \overset{\text{want}}{=} 0$$

define $a_{ip}^k = -\Gamma_{pi}^k\Big|_0$ ✓ □

(lecture 6)

lemma: $T|_x = 0 \iff \exists$ coords x^i near x s.t. $\nabla \partial_i|_x = 0$

proof: ($\Leftarrow$) T tensor

$\longrightarrow$ pick x^i st. $\nabla \frac{\partial}{\partial x^i}\Big|_x = 0$

$$\Rightarrow T(\partial_i, \partial_j) = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i - [\partial_i, \partial_j]$$

$$@x \quad = 0 - 0 - \cancel{(\partial_i \partial_j - \partial_j \partial_i)}^{\to 0}$$

$$= 0$$

($\Rightarrow$) pick (x^i) coords near $x \equiv 0$

$$0 = T(\partial_i, \partial_j)\Big|_0 = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i - \cancel{[\partial_i, \partial_j]}^{0}$$

$$= (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k$$

$$\Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji}$$

define new coords y^i by $x^i = y^i + a^i_{jk} y^j y^k$

a^i_{jk} const, $a^i_{jk} = a^i_{kj}$

$$\hookrightarrow \frac{\partial}{\partial y^i} = \frac{\partial x^p}{\partial y^i} \frac{\partial}{\partial x^p}$$

$$= \frac{\partial}{\partial y^i}\left(y^p + a^p_{jk} y^j y^k \right) \frac{\partial}{\partial x^p}$$

$$= \left(\delta_{ip} + a^p_{jk} \delta_{ij} y^k + a^p_{jk} y^j \delta_{ik} \right) \frac{\partial}{\partial x^p}$$

$$= \left(\delta_{ip} + 2 a^p_{ik} y^k \right) \frac{\partial}{\partial x^p} \underset{@0}{=} \frac{\partial}{\partial x^i}$$

$$\Rightarrow \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}\Big|_0 = \frac{\partial}{\partial y^i}\left(\delta_{jp} + 2a_{jk}^{\,p} y^k\right)\frac{\partial}{\partial x^p}\Big|_{y=0} + \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\Big|_{y=0}$$

$$= 2a_{ij}^{\,p}\frac{\partial}{\partial x^p} + \Gamma_{ij}^{\,p}\frac{\partial}{\partial x^p}$$

$$\overset{\text{want}}{=} 0 \quad \Longrightarrow \quad a_{ij}^{\,p} = -\tfrac{1}{2}\Gamma_{ij}^{\,p}\Big|_0 \qquad \square$$

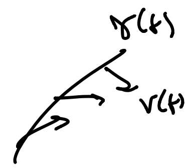
<u>Covariant derivative along curve</u>

M, ∇ = linear connection

$\gamma : I \longrightarrow M$ curve $\quad (I = (a,b) = \text{interval})$

$V(t)$ = vector field on γ i.e. $V(t) \in T_{\gamma(t)}M$, varies smoothly

↳ how to compute t-derivative of V?



first spse $V(t) = \tilde{V}\big|_{\gamma(I)}$ near t

for some $\tilde{V} \in \mathfrak{X}(M)$ $\quad$ (i.e. $\tilde{V}$ = "extension of $V(t)$")

Compute: $\nabla_{\gamma'}\tilde{V}$

in coords: $\gamma(t) = (x^1(t), \ldots, x^n(t)) \longrightarrow \gamma' = \dot{x}^i \partial_i$

$V(t) \equiv \tilde{V}(\gamma(t)) = V^i(t)\,\partial_i\big|_{x(t)}$

$$\Rightarrow \nabla_{\gamma'} \tilde{V} = \nabla_{\gamma'}(V^i \partial_i) =$$

$$= \underline{\gamma'(V^i)}\, \partial_i + V^i\, \underline{\nabla_{\gamma'}} \partial_i$$

$\hookrightarrow \frac{d}{dt} V^i(\gamma(t)) = \frac{d}{dt} V^i(t)$

$$= \dot{V}^i \partial_i + V^i \dot{x}^j \Gamma_{ji}^k \partial_k$$

$$(*) \qquad = \underline{\left(\dot{V}^k(t) + V^i(t)\, \dot{x}^j(t)\, \Gamma_{ji}^k\big|_{\gamma(t)}\right) \partial_k\big|_{\gamma(t)}}$$

<u>depends only</u> on $\tilde{V}\big|_{\gamma(t)} = V(t)$

Can <u>define</u> $\frac{DV}{dt}$ = covariant derivative of $V(t)$ along $\gamma(t)$ by $(*)$

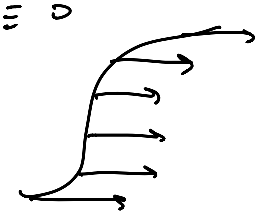
<u>properties:</u> ① (linear/$\mathbb{R}$) $\frac{D}{dt}(aV_1 + bV_2) = a\frac{DV_1}{dt} + b\frac{DV_2}{dt}$ $\quad a,b \in \mathbb{R}$

② (Leibnitz) $\frac{D}{dt}(f(t)V(t)) = \dot{f}V + f\frac{DV}{dt}$

③ (compat) if $\tilde{V} \in \mathfrak{X}(M)$ extension of V near $\gamma(t)$

then $\frac{DV}{dt}\Big|_t = \nabla_{\gamma'}\tilde{V}\Big|_{\gamma(t)}$

<u>Defn</u>: $V(t)$ = parallel along $\gamma(t) \iff \frac{DV}{dt} \equiv 0$



lemma: $\gamma : [a,b] \longrightarrow M$ smooth curve, $v \in T_{\gamma(a)} M$

$\Rightarrow \exists!$ parallel vector field $V(t) : [a,b] \longrightarrow TM$ on γ

s.t. $V(a) = v$

"parallel transport of v along γ"

exercise: connection determined by parallel transport

ODE fact! U open in $\mathbb{R}^N$, $I = (a,b) \subset \mathbb{R}$

$F(t,y) : I \times U \longrightarrow \mathbb{R}^N$ continuous, loc. Lipschitz in y

given $t_0 \in I$, $y_0 \in U$

$\Rightarrow \exists!$ soln $y(t) : J = (a', b') \longrightarrow U$, $J \ni t_0$

to IVP $\begin{cases} \frac{d}{dt} y = F(t, y(t)) \\ y(t_0) = y_0 \end{cases}$

can assume: if $b' < b$ then either $\left. \begin{array}{l} y(t) \to \infty \\ \text{or } d(y(t), \partial U) \to 0 \end{array} \right\} t \to b'$

ditto for a'

note: if $F(t,y) = A(t) y$ $\quad A(t) =$ matrix

then $J = I$ $\quad \left(|y| \leq e^{ct} |y_0| \text{ if } |A(t)| \leq c \right)$

proof of lemma: need $V(t) : [a,b] \longrightarrow TM$ s.t. $V(a) = v$

$\frac{DV}{dt} \equiv 0$

assume $\gamma[a,b] \subset$ coord chart (x^i)

↳ in coords: write $v = v^i \partial_i$, $\gamma(t) = x^i(t)$

want $\begin{cases} V^i(a) = v^i \\ \dot V^k(t) + \dot x^i(t) V^j(t) \Gamma^k_{ij}(x(t)) = 0 \end{cases}$

for $t \in [a,b]$

linear, first order ODE system for $V^i(t)$

$\Rightarrow \exists!$ soln $V^i(t) : [a,b] \to TM$

for general γ, set $T \in [a,b]$ maximal time

s.t. $V(t) : [a,T] \to TM$

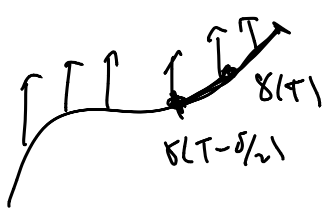
is unique soln to $\begin{cases} V(a) = v \\ \frac{DV}{dt} = 0 \end{cases}$

know $T > a$ by prev. case

now pick coord chart centered @ $\gamma(T)$

$\Rightarrow \gamma(\underbrace{[T-\delta, T+\delta] \cap [a,b]}_{I}) \subset$ chart

$\Rightarrow$ find unique soln $W(t) : I \to TM$



to $\begin{cases} W(T-\delta/2) = V(T-\delta/2) \\ \frac{DW}{dt} \equiv 0 \end{cases}$

uniqueness $\Rightarrow W = V$ on $(T-\delta, T)$

$\Rightarrow$ can extend soln to $[a, T+\delta] \cap [a,b]$

$\Rightarrow T = b$ (else contradict maximality) $\square$

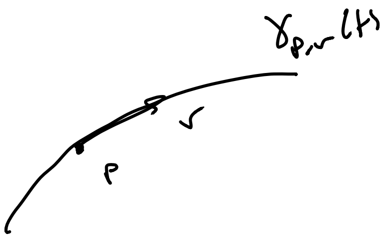
Defn: $\gamma(t)$ = geodesic $\Leftrightarrow \frac{D\gamma'}{dt} \equiv 0$

(lecture 7)

Propn: given any $p \in M$, $v \in T_pM$ $\Rightarrow \exists!$ geodesic $\gamma_{p,v}(t) : I \to M$

I open, $0 \in I$

$\gamma_{p,v}(0) = p$, $\gamma_{p,v}'(0) = v$



proof: (x^i) coords near $p \equiv 0$

want: $x^i(t)$ s.t. $x^i(0) = 0$
$\quad\;\; \shortparallel$ $\qquad\qquad \dot{x}^i(0) = v^i$ where $\underline{v = v^i \partial_i}$
$\quad \gamma(t)$

$$\frac{D\gamma'}{dt} \equiv 0$$
$$= (\dot{x}^k)^{\cdot} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k(x(t))$$
$$= \ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k\Big|_{x(t)}$$
$$= 0$$

consider pair $(x^i(t), v^j(t))$

$\hookrightarrow$ solves IVP
$$\begin{cases} x^i(0) = 0, \quad v^i(0) = v^i \\ \dot{x}^i = v^i \\ \dot{v}^k = \ddot{x}^k = -v^i v^j \Gamma_{ij}^k\Big|_{x(t)} \end{cases}$$

ODE $\Rightarrow \exists!$ soln to IVP for $t \in (-\varepsilon, \varepsilon)$ $\qquad \square$

Rmk: $\lambda \in \mathbb{R}$, $\underline{\gamma_{p,\lambda v}(t)} = \gamma_{p,v}(\lambda t)$ "scaling property"

$\hookrightarrow$ can check: $\alpha(t) = \underline{\gamma_{p,v}(\lambda t)}$

$$\Rightarrow \frac{D\alpha'}{dt} \equiv 0, \quad \alpha(0) = p, \quad \alpha'(0) = \lambda v$$

what does R mean?

Defn: ∇ = trivial in $U \subset M$

$\iff \exists$ basis $F_1 \ldots F_n \in \mathfrak{X}(U)$ s.t. $\nabla F_i \equiv 0$ in U

i.e. $F_1 \cdots F_n$ span T_pM $\forall p \in U$

Propn: curvature $R \equiv 0 \iff \underline{\nabla = \text{locally trivial}}$

$\left(R \equiv 0,\ T \equiv 0 \iff \nabla = \underline{\text{locally trivial}} \text{ by coord fields} \right)$

proof: ($\Leftarrow$) choose $F_1 \ldots F_n \in \mathfrak{X}(M)$ basis near p s.t. $\nabla F_i \equiv 0$

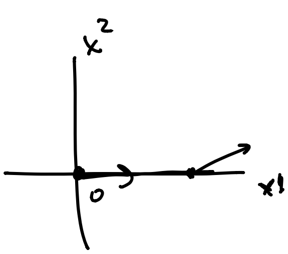
$$\Rightarrow R(F_i, F_j) F_k = \nabla_{F_i} \nabla_{F_j} F_k - \nabla_{F_j} \nabla_{F_i} F_k - \nabla_{[F_i, F_j]} F_k$$

$$= 0$$

$\Rightarrow R = 0$ @ p by tensoriality

($\Rightarrow$) $p \in M$, choose coords $(x^i) : B_1 \to U$ st. $x^i(0) = p$

build $F_1 \ldots F_n$ on B_1 s.t. $\nabla F_i \equiv 0$



$F_i(0) = \partial_i$

$F_i(x^1, 0, 0, \ldots 0)$ = parallel transport of $F_i(0)$ along x^1-curve

$F_i(\underline{x^1, x^2}, 0, \ldots)$ = parallel transport $F_i(x^1, 0, 0, \ldots 0)$ along x^2-curve

⋮

etc.

claim: $\nabla_{\partial_1} F_i = \nabla_{\partial_2} F_i = 0$ on $(x^1, x^2, 0, 0, \ldots 0)$

|

by construction, $\nabla_{\partial_2} F_i \big|_{(x^1, x^2, 0, \dots, 0)} = 0$ (as parallel transport)

now $\nabla_{\partial_2} \nabla_{\partial_1} F_i = \nabla_{\partial_2} \nabla_{\partial_1} F_i - \nabla_{\partial_1} \nabla_{\partial_2} F_i - \nabla_{\underbrace{[\partial_2, \partial_1]}_{=0}} F_i$

$$+ \nabla_{\partial_1} \nabla_{\partial_2} F_i$$

$$= R(\partial_2, \partial_1) F_i + \nabla_{\partial_1} \nabla_{\partial_2} F_i$$

$$= 0 + \nabla_{\partial_1} \nabla_{\partial_2} F_i$$

$$= 0 \quad \text{on } (x^1, x^2, 0, 0 \dots 0)$$

and $\nabla_{\partial_1} F_i \big|_{(x^1, 0, 0, \dots, 0)} = 0$

$\Rightarrow \nabla_{\partial_1} F_i \big|_{(x^1, x^2, 0 \dots, 0)} = 0$ by uniqueness of parallel transport

same method gives $\nabla_1 F_i = \nabla_2 F_i = \nabla_3 F_i = 0$ on $(x^1, x^2, x^3, 0, 0, \dots, 0)$

repeat to get $\nabla_{\partial_i} F_j = 0$ for all $(x^1, x^2, \dots, x^n) \in B$. $\square$

if $T \equiv 0$ also, can get $F_i = \dfrac{\partial}{\partial y^i}$

↳ b/c $[F_i, F_j] = \nabla_{F_i} F_j - \nabla_{F_j} F_i = 0$

$\|$ Frobenius $\Rightarrow$ can integrate F:

Levi-Civita connection

(M, g), ∇ = linear connection = metric compatible $\Leftrightarrow$ $\nabla g \equiv 0$

$$\Leftrightarrow X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

thm: $\exists!$ linear connection which is metric-compatible, torsion-free

"Levi-Civita connection"

Ex: if $M \subset \mathbb{R}^{n+k}$ embedded submanifold

$\hookrightarrow g_M = g_{eucl}|_{TM}$, $\nabla^M = \pi_{TM} D$

i.e. $\nabla^M_X Y = \pi_{TM}(D_X Y)$

then $\nabla^M g_M = 0$, ∇^M torsion-free

proof: find expression for ∇

choose coords $x^i \to$ need to find $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}$

$$\left(\text{then } \begin{matrix} X = X^i\partial_i \\ Y = Y^j\partial_j \end{matrix} \Rightarrow \nabla_X Y = X^i\partial_i(Y^j)\partial_j + X^iY^j\Gamma^k_{ij}\partial_k\right)$$

$$\partial_i\langle\partial_j, \partial_k\rangle = \partial_i g_{jk} = \langle\Gamma^p_{ij}\partial_p, \partial_k\rangle + \langle\partial_j, \Gamma^p_{ik}\partial_p\rangle$$

$$\partial_i g_{jk} = \Gamma^p_{ij} g_{pk} + \Gamma^p_{ik} g_{jp}$$

$$+ \quad \partial_j g_{ki} = \Gamma^p_{jk} g_{pi} + \Gamma^p_{ij} g_{kp}$$

$$- \quad \partial_k g_{ij} = \Gamma^p_{ik} g_{pj} + \Gamma^p_{jk} g_{ip}$$

torsion free

$$\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma_{ij}^{p} g_{pk} + 0$$

so $$\Gamma_{ij}^{k} = \frac{1}{2} g^{kp} \left(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij} \right)$$ $\square$

coord free expression: $X, Y, Z \in \mathfrak{X}(M)$

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle + \langle [X,Y], Z\rangle \Big)$$

lecture 8

(M^n, g) Riemannian, ∇ = LC connection (ie, $\nabla g \equiv 0$, $T \equiv 0$)

thm: $R \equiv 0 \iff (M, g)$ loc. isometric to $(\mathbb{R}^n, g_{eucl})$

proof: $\Leftarrow$ $p \in M$, choose coords (x^i) near p s.t. $g_{ij} = \delta_{ij}$

$\Rightarrow \partial_k g_{ij} = 0$

$\Rightarrow \Gamma_{ij}^{k} = 0 \Rightarrow \nabla \partial_i = 0$

$\Rightarrow R(\partial_i, \partial_j)\partial_k = 0$

$\Rightarrow R \equiv 0$ (tensoriality)

$\Rightarrow$ by prev. propn, $\exists$ basis $F_1 \dots F_n \in \mathfrak{X}(M)$ near p s.t. $\nabla F_i \equiv 0$

replace F_i with $a_i^j F_j$ (a_i^j const)

$\Rightarrow$ can assume $\langle F_i, F_j \rangle = \delta_{ij}$ @ p

now $[F_i, F_j] = \nabla_{F_i} F_j - \nabla_{F_j} F_i = 0$

$\Rightarrow$ (Frobenius)

$\exists$ smooth $\varphi(y^1 \dots y^n, x^1 \dots x^n) : B_\varepsilon(0) \times U \longrightarrow M$, with $B_\varepsilon(0) \subset \mathbb{R}^n$, $U \subset M$, $p \in U$

s.t. $\dfrac{\partial \varphi}{\partial y^i} = F_i(\varphi)$

$D_y \varphi\big|_{(0,p)} : T_0 \mathbb{R}^n \longrightarrow T_p M$ non-singular

(since $\dfrac{\partial \varphi}{\partial y^i}\Big|_{(0,p)} = F_i(p) = g\text{-ON}$)

(inverse f^n thm) $\Rightarrow \varphi$ smooth diff. $B_\delta(0) \longrightarrow M$
onto image $\quad 0 \longmapsto p$

$\Rightarrow$ gives coord chart near p, $\dfrac{\partial}{\partial y^i} = F_i$

$g_{ij}(0) = \langle F_i(p), F_j(p) \rangle = \delta_{ij}$

and $\partial_k g_{ij} = \left\langle \nabla_{\frac{\partial}{\partial y^k}} \dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j} \right\rangle + \left\langle \dfrac{\partial}{\partial y^i}, \nabla_{\frac{\partial}{\partial y^k}} \dfrac{\partial}{\partial y^j} \right\rangle$

$= \langle \nabla_k F_i, F_j \rangle + \langle F_i, \nabla_k F_j \rangle$

$= 0$

$\Rightarrow g_{ij}(y) = g_{ij}(0) = \delta_{ij}$ $\qquad \square$

Riemannian geodesics, exponential map

$\gamma(t)$ = Riem. geodesic $\left(\iff \frac{D\gamma'}{dt} \equiv \nabla_{\gamma'}\gamma' = 0, \quad \nabla = \text{LC connect}\right)$

Rmk: speed $|\gamma'| = \text{const}$

check: $\frac{d}{dt}\left(|\gamma'|^2\right) = \frac{d}{dt}\langle \gamma', \gamma'\rangle = 2\langle \cancel{\nabla_{\gamma'}\gamma'}^{\,0}, \gamma'\rangle$ $\square$

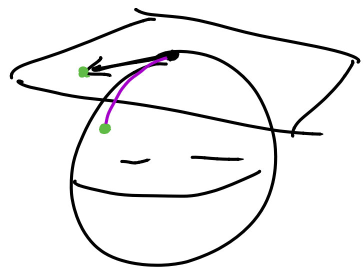
$= 0$

let $\gamma_{p,v}(t)$ = geodesic s.t. $\gamma_{p,v}(0) = p$, $\gamma_{p,v}'(0) = v$

$$\mathcal{E} = \{ (p,v) \in TM : \gamma_{p,v} \text{ exists on } [0,1] \}$$

define $\exp : \mathcal{E} \subset TM \longrightarrow M$

$(p,v) \longmapsto \gamma_{p,v}(1)$



$\exp_p : \mathcal{E} \cap T_pM \longrightarrow M$

Ex: $M = S^n \subset \mathbb{R}^{n+1}$, $p = e_{n+1} \implies T_pS^n = \mathbb{R}^n \times \{0\}$

$v \in T_pM = \mathbb{R}^n \times \{0\}$

$\gamma_{p,v}(t) = e_{n+1}\cos(|v|t) + \frac{v}{|v|}\sin(|v|t)$

$\exp_p(v) = e_{n+1}\cos(|v|) + \frac{v}{|v|}\sin(|v|)$

Rmk: "exp" comes from Lie groups

$\hookrightarrow$ if $G \subset GL_n(\mathbb{R}^n)$, $A \mapsto e^A$

gives map (Lie algebra of $G = T_{Id}G$)

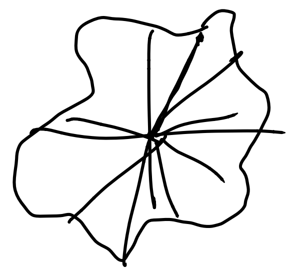
$\updownarrow$

G

lemma: $\mathcal{E}$ = open in TM, contains $M \times \{0\}$, $\mathcal{E}_p$ = star-shaped
and $\exp : \mathcal{E} \to M$ smooth

$\mathcal{E}_p = \mathcal{E} \cap T_pM$

proof: (essentially ODE fact...)



reconsider geodesic equation

$\gamma(t) = x^i(t)$ = geodesic

$\Updownarrow$

$(x^i(t), v^i(t))$ solves ODE $\quad \dot{x}^k = v^k$, $\quad \dot{v}^k = -v^i v^j \Gamma_{ij}^k(x)$

$\Updownarrow$

$(x^i(t), v^i(t))$ = integral curve for vector field
$G(x,v) = (v^k, -v^i v^j \Gamma_{ij}^k(x))$

$\left[\begin{array}{l} (x^i) \text{ coords on } M \text{ near } p \\ (x^i, v^j \frac{\partial}{\partial x^j}) \text{ coords on } TM \text{ near } (p,0) \\ (x^i, v^j \frac{\partial}{\partial x^j}, a^k \frac{\partial}{\partial x^k} + b^\ell \frac{\partial}{\partial v^\ell}) \text{ coords on } T(TM) \text{ near } (p,0,0) \end{array}\right.$

define G = smooth vector field on TM

$G(x,v) = (v^k, -v^i v^j \Gamma_{ij}^k(x))$

$\hookrightarrow \quad G(x,v) = \frac{d}{dt}\Big|_{t=0} (\gamma_{x,v}(t), \gamma_{x,v}'(t))$

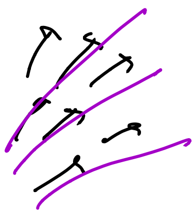
(so G well-defined)

then: $\gamma(t)$ = geodesic in $M \iff (\gamma(t), \gamma'(t))$ = int. curve of G

ODE $\Rightarrow \exists$ open set $\mathcal{O} \subset \mathbb{R} \times TM$
$\qquad\qquad\qquad\qquad \cup$
$\qquad\qquad\qquad\quad \{0\} \times TM$

s.t. flow $\varphi_t : \mathcal{O} \to TM$ of G exists, smooth on $\mathcal{O}$

i.e. $\dfrac{\partial \varphi_t(x,v)}{\partial t} = G(\varphi_t(x,v))$



"geodesic flow"

$\gamma_{x,0}(0) = x$ defined $\forall t \Rightarrow M \times \{0\} \subset \mathcal{E}$

if $(p,v) \in \mathcal{E} \Rightarrow (1,p,v) \in \mathcal{O}$

$\Rightarrow \exists\ \varepsilon > 0$, nbd $\underset{(p,v)}{\overset{}{U}} \subset TM$ s.t. $(1-\varepsilon, 1+\varepsilon) \times U \subset \mathcal{O}$

$\Rightarrow U \subset \mathcal{E} \quad \Rightarrow \mathcal{E}$ open

$\mathcal{E}_p$ star-shaped: $(p,v) \in \mathcal{E} \iff \gamma_{p,v}(t)$ exists on $[0,1]$

$\iff \gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$

exist for $0 \le t \le \frac{1}{\lambda}$

if $\lambda \in [0,1] \Rightarrow (p, \lambda v) \in \mathcal{E}$

$\exp(p,v) = \varphi_{t=1}(p,v)$ = smooth $\square$

<u>recall</u>: $\exp_p : T_pM \cap \mathcal{E} \to M$

Claim: $D\exp_p\big|_0 : T_0(T_pM) = T_pM \longrightarrow T_pM$

$= \text{id map}$

proof: $v \in T_pM \Rightarrow D\exp_p\big|_0(v)$

$$= \frac{d}{dt}\Big|_{t=0} \exp_p(tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{p,tv}(1)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{p,v}(t) = \gamma'_{p,v}(0) = v \quad \square$$

$\Rightarrow \exp_p = $ loc. diffeom near 0

if $e_1 \dots e_n = g$-ONl basis of T_pM

can define $(x^1, \dots, x^n) \longmapsto \exp_p(\sum_i x^i e_i)$ coords near p

"normal coordinates near p"

$\hookrightarrow$ metric $g_{ij}\big|_x = \langle D\exp_p\big|_x(e_i), D\exp_p\big|_x(e_j)\rangle$

in normal coords: (1) $t \mapsto tv$ = geodesics

(2) $g_{ij} = \delta_{ij} + O(|x|^2)$ (g eucl. to first order)

(3) + other stuff...

proof of (1): $x^i(t) = tv^i$ = geodesic in normal coords satisfy: $\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma^k_{ij}(x(t)) = 0$

$$0 + v^i v^j \Gamma^k_{ij}(tv) = 0$$

@ $t=0 \Rightarrow \quad v^i v^j \Gamma^k_{ij}(0) = 0 \quad \forall v \in \mathbb{R}^n$

and $\Gamma^k_{ij} = \Gamma^k_{ji} \Rightarrow \Gamma^k_{ij}(0) = 0$

$$\Rightarrow \partial_k g_{ij}\big|_0 = \partial_k \langle \partial_i, \partial_j \rangle$$
$$= \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle$$
$$= \Gamma^p_{ki} g_{pj} + \Gamma^p_{kj} g_{pi} = 0 \quad @\ 0$$

and $g_{ij} = \delta_{ij}$ since $D\exp_p\big|_0 = I$ $\square$

other nice properties for $\exp_p$ ... (later)

also later: control size of normal neighborhood by topology of (M,g) and curvature

for now: "loc. uniform normal neighborhood size"

<u>Lemma:</u> for $p \in M \Rightarrow \exists$ nhd $\underset{p}{\underset{\in}{U}} \subset M$, and $\varepsilon > 0$

s.t. $\exp_q\big|_{B_\varepsilon(0)}$ = diffeo $\forall q \in U$

<u>proof:</u> define $F: TM \to M \times M$

$$(p,v) \mapsto (p, \exp_p(v))$$

ETS: F = loc. diffeo near $(p,0) \in TM$

Γ if $\exists$ nhd $\underset{\in}{U} \subset M$, $\varepsilon > 0$ s.t. $F\big|$ = diffeo

p $\qquad\qquad$ $|_{U \times B_\varepsilon(0)}$

$$\Rightarrow F\big|_{\{q\}\times B_\varepsilon(0)}(v) = (q, \exp_q(v))$$

$$: B_\varepsilon(0) \cap T_q M \longrightarrow \{q\} \times M \quad = \text{diffeo}$$

WTS: $DF\big|_{(p,0)} : \underline{T_{(p,0)}}(TM) \longrightarrow T_pM \times T_pM$ = non-singular

(then use inverse f^n thm)

coords (x^i) near p

identify $T_{(p,0)}(TM) \longleftrightarrow T_pM \times T_pM$

$$\underline{a^i \frac{\partial}{\partial x^i} + b^j \frac{\partial}{\partial v^j} \longleftrightarrow \left(a^i \frac{\partial}{\partial x^i}, b^j \frac{\partial}{\partial x^j}\right)}$$

now $$DF\big|_{(p,0)}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial F}{\partial x^i}\bigg|_{(p,0)} = \frac{\partial}{\partial x^i}(x, \exp_x(0))$$

$$= \frac{\partial}{\partial x^i}(x, x)$$

$$= \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right)$$

$$DF\big|_{(p,0)}\left(\frac{\partial}{\partial v^i}\right) = \frac{\partial}{\partial v^i}(p, \exp_p(v))$$

$$= \left(0, D\exp_p\big|_0\left(\frac{\partial}{\partial v^i} \equiv \frac{\partial}{\partial x^i}\right)\right)$$

$$= \left(0, \frac{\partial}{\partial x^i} \equiv \frac{\partial}{\partial v^i}\right)$$

$$\frac{\partial}{\partial x^i} \qquad \frac{\partial}{\partial v^i}$$

$$DF\Big|_{(p,0)} = \boxed{\begin{array}{c|c} I & 0 \\ \hline * & I \end{array}} \text{ is non-singular} \quad \checkmark \qquad \square$$

first variation of length

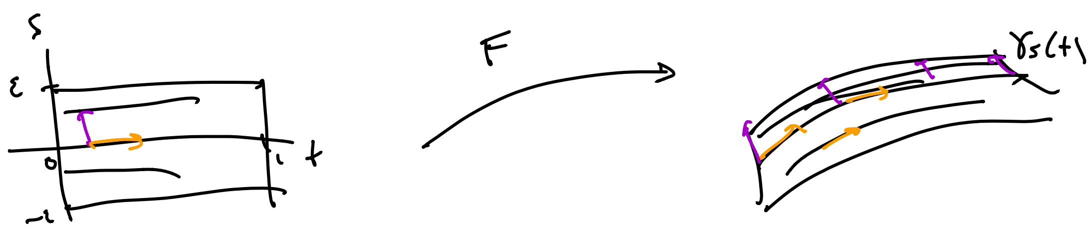
$\gamma(t) : [0,1] \longrightarrow M$ = smooth curve with const speed i.e. $|\gamma'(t)| = |\gamma'(0)|$

$$\text{length } \gamma = L\gamma = \int_0^1 |\gamma'(t)|\, dt \quad (= |\gamma'(0)|)$$

variation of γ $= \gamma_s(t) \equiv F(s,t) : (-\varepsilon, \varepsilon) \times [0,1] \xrightarrow{\text{smooth}} M$

s.t. $\gamma_0 = \gamma$

= 1-parameter family of curves



$$V(t) = DF\Big|_{(t,0)}\left(\frac{\partial}{\partial s}\right) = \underline{\text{infinitesimal variation}}$$

first variation of length of γ w.r.t. V $= \delta L(\gamma)[V]$

$$= \frac{d}{ds}\Big|_{s=0} L\gamma_s$$

claim: $\dfrac{D}{ds}\dfrac{\partial F}{\partial t} = \dfrac{D}{dt}\dfrac{\partial F}{\partial s}$ vectors ie. $\dfrac{\partial F}{\partial s} = DF\left(\dfrac{\partial}{\partial s}\right)$

$\dfrac{\partial F}{\partial t} = DF\left(\dfrac{\partial}{\partial t}\right)$

proof: in coords: $\frac{\partial F}{\partial t} = \frac{\partial F^k}{\partial t} \partial_k$ etc.. velocity vector of curve

$$\left(\frac{D}{ds}\frac{\partial F}{\partial t}\right)^k = \frac{\partial}{\partial s}\frac{\partial F^k}{\partial t} + \frac{\partial F^i}{\partial s}\frac{\partial F^j}{\partial t}\Gamma^k_{ij}(s,t)$$

$$= \frac{\partial^2 F^k}{\partial s \partial t} + \frac{\partial F^i}{\partial t}\frac{\partial F^j}{\partial s}\Gamma^k_{ij}(s,t)$$

$$= \frac{\partial}{\partial t}\frac{\partial F^k}{\partial s} + \frac{\partial F^i}{\partial t}\frac{\partial F^j}{\partial s}\Gamma^k_{ij}(s,t)$$

$$= \left(\frac{D}{dt}\frac{\partial F}{\partial s}\right)^k \qquad \square$$

alternatively (let's assume F immersion)

$$\frac{D}{dt}\frac{\partial F}{\partial s} = \nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial s} = \nabla_{\frac{\partial F}{\partial s}}\frac{\partial F}{\partial t} + \left[\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right]$$

$$= \frac{D}{ds}\frac{\partial F}{\partial t} + \underbrace{\left[DF(\partial_t), DF(\partial_s)\right]}_{\substack{\shortparallel \\ DF[\partial_s, \partial_t] = 0}}$$

thm: $\frac{d}{ds}\Big|_{s=0} L\gamma_s = \frac{1}{L\gamma} \langle V(t), \gamma'(t)\rangle \Big|_{t=0}^{1} - \frac{1}{L\gamma}\int_0^1 \langle V(t), \frac{D\gamma'}{dt}\rangle \, dt$

proof: $\frac{d}{ds}\Big|_{s=0} L\gamma_s = \frac{d}{ds}\Big|_{s=0} \int_0^1 \langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\rangle^{1/2} \, dt$

$$= \int_0^1 \frac{1}{|\gamma'|} \langle \frac{D}{ds}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\rangle \, dt$$

const speed $\longrightarrow \displaystyle = \frac{1}{|\gamma'(0)|}\int_0^1 \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right\rangle dt$

$$= \frac{1}{L\gamma}\int_0^1 \underline{\frac{d}{dt}\left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right\rangle} - \left\langle \frac{\partial F}{\partial s}, \frac{D}{dt}\frac{\partial F}{\partial t}\right\rangle dt$$

$$= \frac{1}{L\gamma}\left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right\rangle\Big|_0^1 - \frac{1}{L\gamma}\int_0^1 \left\langle \frac{\partial F}{\partial s}, \frac{D}{dt}\frac{\partial F}{\partial t}\right\rangle dt$$

$V = \frac{\partial F}{\partial s}\Big|_{s=0}$

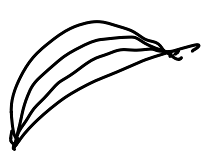
$\gamma' = \frac{\partial F}{\partial t}\Big|_{s=0}$

$$\underline{\frac{d}{ds}\Big|_{s=0} L\gamma_s = \frac{1}{L\gamma}\langle V, \gamma'\rangle\Big|_0^1 - \frac{1}{L\gamma}\int_0^1 \left\langle V, \frac{D\gamma'}{dt}\right\rangle dt} \quad \square$$

<u>Consequences:</u> ① if γ const speed, $\frac{d}{ds}\Big|_{s=0} L = 0$ $\forall$ variations $V(0)=V(1)=0$

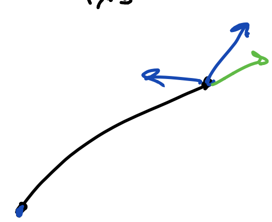
$\Updownarrow$

γ = geodesic

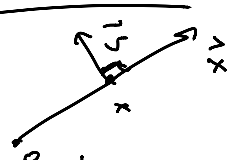


② if γ = geodesic

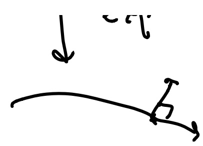
then $\displaystyle \frac{d}{ds}\Big|_{s=0} L\gamma_s = \frac{1}{L\gamma}\big(\langle V(1), \gamma'(1)\rangle - \langle V(0), \gamma'(0)\rangle\big)$



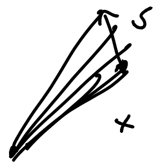
<u>lemma (Gauss' lemma):</u> $\underline{\langle D\exp_p|_x(x), D\exp_p|_x(v)\rangle = \langle x, v\rangle}$ $\quad \underline{\forall x, v \in T_pM \cap \varepsilon}$



"$\exp_p$ = radial isometry"



proof: Consider variation $F(s,t) = \exp_p(t(x+sv))$



$$\frac{\partial F}{\partial s}\Big|_{s=0} = D\exp_p\big|_{tx}(tv) = \underline{V} \quad \text{variation vector}$$

$$\frac{\partial F}{\partial t}\Big|_{s=0} = D\exp_p\big|_{tx}(x) = \gamma' \quad \text{velocity vector}$$

each $\gamma_s(t) = F(s,t)$ = geodesic with $\gamma_s'(0) = x+sv$

$$\Rightarrow L\gamma_s = \int_0^1 |\gamma_s'(t)|\,dt = \underline{|x+sv|} = \langle x+sv, x+sv\rangle^{1/2}$$

$$\Rightarrow \frac{d}{ds}\Big|_{s=0} L\gamma_s = \frac{\langle x, v\rangle}{|x|}$$

$$= \frac{1}{L\gamma_0}\left(\langle V(1), \gamma'(1)\rangle - \langle V(0), \gamma'(0)\rangle\right)$$

$$= \frac{1}{|x|}\langle D\exp_p|_x(v), D\exp_p|_x(x)\rangle + 0 \quad \square$$

next time:

$$r = \frac{x}{|x|}$$

in normal coords $g|_x(v,w) = \langle D\exp_p|_x(\underline{v}), D\exp_p|_x(\underline{w})\rangle$

$$= (\exp_p)^* g_M$$

"generalized polar coords" $(r,\theta) \in \mathbb{R}\times S^{n-1}$

T

$$\downarrow$$
$$\exp_p(r\theta)$$

$$g = dr^2 + r^2 f_{ij}(r,\theta)\, d\theta^i d\theta^j$$

no cross terms $dr\, d\theta^i$

"generalized polar coords"

identify $(T_pM, g|_p) \cong (\mathbb{R}^n, g_{eucl})$ via some isometry

↳ define $(r,\theta) \in (0,\varepsilon] \times S^{n-1} \longmapsto \exp_p(r\theta) \in M$

(for ε small so that $B_\varepsilon \subset \mathcal{E}_p$)

smooth metric on S^{n-1}

metric $g = (\exp_p^*) g_M = \underline{dr^2 + r^2 f_{ij}(r,\theta)\, d\theta^i d\theta^j}$

no cross terms $dr\, d\theta^i$

check: $g = \underline{g(\partial_r, \partial_r)}\, dr^2 + 2g\underline{(\partial_r, \partial_{\theta^i})}\, dr\, d\theta^i + \underline{g(\partial_{\theta^i}, \partial_{\theta^j})}\, d\theta^i d\theta^j$

↳ $\partial_r = D\exp_p|_{r\theta}(\theta)$, $\partial_{\theta^i} = D\exp_p|_{r\theta}(r\partial_{\theta^i})$

so $g(\partial_r, \partial_r) = \langle D\exp_p|_{r\theta}(\theta), D\exp_p|_{r\theta}(\theta)\rangle|_{\exp_p(r\theta)}$

$$\text{GL} = \langle \theta, \theta \rangle|_p$$
$$= 1$$

✓

$$g(\partial_r, \partial_{\theta^i}) = \langle D\exp_p|_{r\theta}(\theta), D\exp_p|_{r\theta}(r\partial_{\theta^i}) \rangle$$
$$= \langle \theta, r\partial_{\theta^i} \rangle$$
$$= 0$$

$$g(\partial_{\theta^i}, \partial_{\theta^j}) = \langle D\exp_p|_{r\theta}(r\partial_{\theta^i}), D\exp_p|_{r\theta}(r\partial_{\theta^j}) \rangle$$
$$= r^2 \underbrace{\langle D\exp_p|_{r\theta}(\partial_{\theta^i}), D\exp_p|_{r\theta}(\partial_{\theta^j}) \rangle}_{\text{smooth fn of } r,\theta}$$

[compare: in $(\mathbb{R}^n, g_{eucl})$: $g = dr^2 + r^2 g_{sphere}$

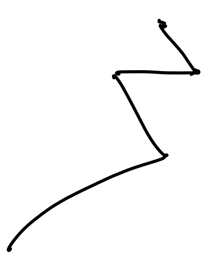
Metric geometry of (M, g) = Riemannian mfld

Defn: smooth curve $\gamma : [0,1] \to M$ = regular if $\gamma'(t) \neq 0$

γ = piecewise regular if γ continuous

and $\exists\ 0 = t_0 < t_1 < \cdots < t_k = 1$

s.t. $\gamma|_{[t_i, t_{i+1}]}$ = regular



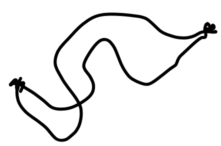
note: γ = p.w. regular $\longrightarrow$ $s(t) = \int_0^1 |\gamma'(t)|\,dt$ strictly $\nearrow$

smooth on each $[t_i, t_{i+1}]$

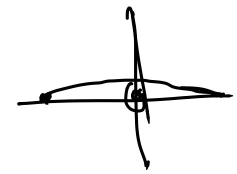
$\Rightarrow$ can reparam γ s.t. $|\gamma'(t)| = \text{const}$

$p, q \in M$, <u>define</u> $d_g(p,q) = \inf \left\{ L\gamma : \begin{array}{l} \gamma : [0,1] \to M \\ = \text{p.w. regular} \quad \gamma(0) = p \\ \qquad\qquad\qquad\quad \gamma(1) = q \end{array} \right\}$

(or γ const)



<u>note:</u> not necessarily realized e.g. $\mathbb{R}^2 \setminus \{0\}$



<u>thm:</u> (M, d_g) = metric space, induced topology = manifold topology

<u>proof:</u> $d(p,q) = d(q,p)$, $d(p,q) \le d(p,s) + d(s,q)$ trivial

WTS: $d(p,q) = 0 \Rightarrow p = q$

take coords $(x^i) : \underline{B_3} \to M$ s.t. $x^i(0) = p$

$\hookrightarrow$ g_{ij} smooth $\Rightarrow$ on B_2, $\underline{\frac{1}{\Lambda^2}\delta_{ij} \le g_{ij} \le \Lambda^2 \delta_{ij}}$

i.e. $\frac{1}{\Lambda^2}\sum v^{i\,2} \le g\big|_x(v,v) \le \Lambda^2 \sum v^{i\,2}$

$\forall x \in B_2$

$\forall v \in \mathbb{R}^n$

$\Rightarrow$ if $\gamma : [0,1] \to B_2 \subset M$

then $L_g\gamma = \int_0^1 g(\gamma',\gamma')^{1/2}\,dt \le \Lambda \int_0^1 \delta(\gamma',\gamma')^{1/2}\,dt$

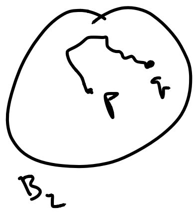
$= \Lambda L_\delta \gamma$

& $\ge \frac{1}{\Lambda} L_\delta \gamma \ge \frac{1}{\Lambda}|\gamma(0) - \gamma(1)|$

$d(p,q) = 0$
$\shortparallel$

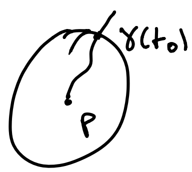
now choose $\gamma : [0,1] \longrightarrow M$ p.w. reg. $\gamma(0) = p,\ \gamma(1) = q$

$L_g \gamma < \varepsilon$



Case 1: $\gamma[0,1] \subset B_2$ $(\Rightarrow q \in B_2)$

$$\Rightarrow \varepsilon > L_g \gamma \geq \frac{1}{\Lambda} |p - q|$$



case 2: $\gamma[0,1] \not\subset B_2$

t_0 first time (>0) s.t. $\gamma(t) \notin B_2$

$$\Rightarrow \varepsilon > L_g \gamma \geq L_g \gamma\big|_{[0,t_0]}$$

$$\geq \frac{1}{\Lambda} |p - \gamma(t_0)|$$

$$= \frac{2}{\Lambda} \qquad \text{↯ if } \varepsilon \text{ small}$$

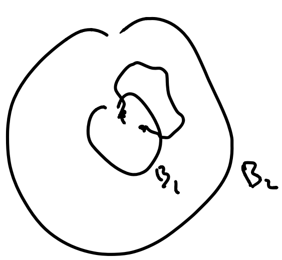
$\Longrightarrow p = q$ ✓

manifold topology = induced charts

ETS: (B_1, d_g) has same topology as $(B_1$, eucl. metric$)$

$q, q' \in B_1$, pick p.v. reg. $\gamma : [0,1] \longrightarrow M$, $\gamma(0) = q$

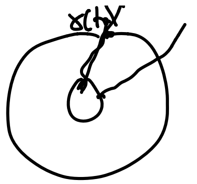
$\gamma(1) = q'$



Case 1: $\gamma[0,1] \subset B_2$

$$\Rightarrow L_g \gamma \geq L_\delta \gamma \geq \frac{1}{\Lambda} |q - q'|$$

Case 2: $\gamma[0,1] \not\subset B_2$, pick t_0 first time s.t. $\gamma(t) \notin B_2$



$$\Rightarrow L_g\gamma \geq L_g \gamma\big|_{[0,t_0]} \geq \frac{1}{\Lambda}\,|q-\gamma(t_0)| \geq \underline{\frac{1}{\Lambda}} \geq \frac{1}{2\Lambda}\,|q-q'|$$

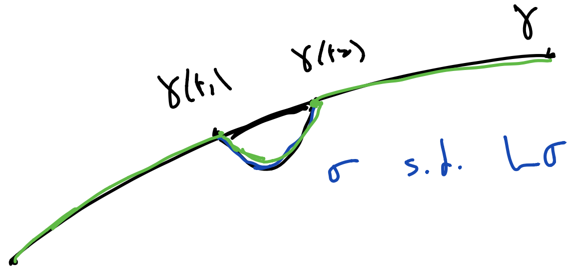
OTOH: $\gamma(t) = q + t(q'-q)$

$$|q-q'| = L_\delta\gamma \geq \frac{1}{\Lambda} L_g\gamma \geq \frac{1}{\Lambda} d_g(q,q')$$

$\boxed{\text{So}}$ $\underline{\frac{1}{2\Lambda}|q-q'| \leq d_g(q,q') \leq 2\Lambda|q-q'|}$ ✓ $\square$

Defn: a p.w. reg $\gamma:[0,1]\to M$ = $\underline{\text{minimizing}}$ if $d_g(\gamma(1),\gamma(0)) = L\gamma$

Note: γ minz $\Longrightarrow$ $\gamma\big|_{[t_1,t_2]}$ minz $\forall\ 0\leq t_1<t_2\leq 1$



if $L\gamma\big|_{[t_1,t_2]} < d(\gamma(t_1), \gamma(t_2))$

σ s.t. $L\sigma < L\gamma\big|_{[t_1,t_2]}$

$\tilde\gamma$ p.w. reg. and $\tilde\gamma(0)=\gamma(0)$, $\tilde\gamma(1)=\gamma(1)$
but $L\tilde\gamma < L\gamma = d(\gamma(0),\gamma(1)) \leq L\tilde\gamma$ ↯

(converse false)

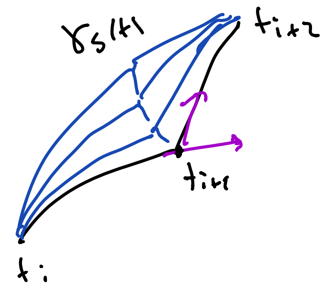
Prop^n: γ p.w. reg curve, const. speed, minz $\Rightarrow$ γ = geodesic

proof: idea: look at variations of Length

↳ $\delta L[V] \geq 0$ $\forall$ variations V fixing ends

$\Rightarrow \delta L[V] = 0$

$0 = t_0 < t_1 < \cdots < t_k = 1$



take variation

$\gamma_s(t) \equiv F(s,t)$ of p.w. reg. curves

s.t. $F(s,\cdot)\big|_{[t_i, t_{i+1}]}$ smooth

F smooth in s, contin in t

$F(s,0) = \gamma(0)$

$F(s,1) = \gamma(1)$

$F(0,t) = \gamma(t)$

$V(t) = \dfrac{\partial F}{\partial s}$

$$\frac{d}{ds}\Big|_{s=0} L\gamma_s = -\sum_i \frac{1}{L}\int_{t_i}^{t_{i+1}} \langle V, \underline{\frac{D\gamma'}{dt}} \rangle + \sum_i \frac{1}{L} \langle V(t_i), \underline{\gamma'(t_i^+) - \gamma'(t_i^-)} \rangle$$

$\quad\quad \shortparallel$

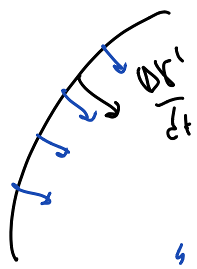
$\quad\quad 0 \quad \forall V$

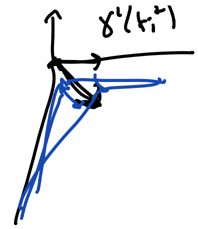
$V \approx \dfrac{D\gamma'}{dt} \Rightarrow \underline{\dfrac{D\gamma'}{dt} = 0 \text{ away from } t_i}$

$V \approx \gamma'(t_i^+) - \gamma'(t_i^-) \Rightarrow \underline{\gamma'(t_i^+) = \gamma'(t_i^-)}$

$\gamma'(t_i^-)$

picture:





" $\frac{D\gamma'}{dt}$ = curvature (of unit speed)

= negative L^2 gradient of length"

if $\sigma : (-\delta, \delta) \to M$ geodesic s.t. $\sigma(0) = \gamma(t_i)$

$\sigma'(0) = \gamma'(t_i)$

(uniqueness) $\Rightarrow \sigma(t) = \gamma(t_i + t)$, t small

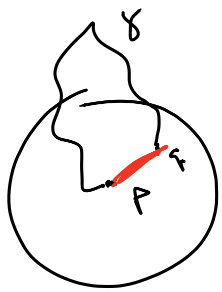
$\Rightarrow \gamma$ smooth , $\frac{D\gamma'}{dt} = 0$ □

propn: let $U = \exp_p(B_\varepsilon(0))$ = normal neighborhood @ p

$q \in U$, $\gamma : [0,1] \to M$ = p.w. regular curve, const speed

s.t. $\gamma(0) = p$

$\gamma(1) = q$



then $L\gamma \geq r(q) \equiv |\exp_p^{-1}(q)|$

(=) $\Leftrightarrow \gamma(t) = \exp_p(tv)$

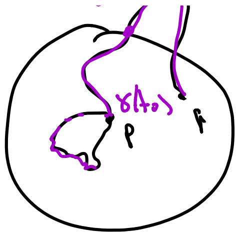
for $v = \exp_p^{-1}(q)$

"geodesic rays loc. uniquely realize distance"

proof: wlog γ has const speed ($\Rightarrow L\gamma = |\gamma'(0)|$), $p \neq q$

choose t_0 s.t. $\gamma(t_0, 1] \subset M \setminus \{p\}$, $\gamma(t_0) = p$

$\gamma(t_i)$



choose t_1 s.t. $\gamma(t_0, t_1) \subset U\setminus\{p\}$

and either $\gamma(t_1) \in \partial U$ or $\gamma(t_1) = q$

now: $\gamma(t_0, t_1) \subset U \setminus \{p\}$

$\Rightarrow$ can use generalized polar coords

$$(0,\varepsilon) \times S^{n-1} \longrightarrow U\setminus\{p\}$$

$$(r, \theta^i) \longmapsto \underline{\exp_p(r\theta)}$$

$\Rightarrow$ $\gamma(t) = (r(t), \theta^i(t))$ for $t \in (t_0, t_1)$

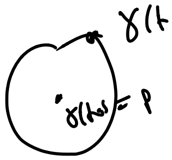
$$\gamma'(t) = \dot{r}\,\partial_r + \dot{\theta}^i \partial_{\theta^i} \qquad g_{ij} = dr^2 + r^2 f_{ij}\, d\theta^i d\theta^j$$

$$\longrightarrow \quad \underline{|\gamma'|^2 = \dot{r}^2} + \underbrace{r^2 f_{ij}\,\dot{\theta}^i \dot{\theta}^j}_{\text{metric on sphere}}$$

$$\underline{\geq \dot{r}^2}$$

<u>So:</u> $L\gamma \geq L\gamma\big|_{(t_0,t_1)} = \lim_{\delta\to 0} \int_{t_0+\delta}^{t_1-\delta} |\gamma'|\, dt$

$$\geq \lim_{\delta \to 0} \int_{t_0+\delta}^{t_1-\delta} \dot{r}\, dt \qquad \frac{d}{dt} r(\gamma(t))$$



$$= \lim_{\delta\to 0} r(\gamma(t_1 - \delta)) - \underbrace{r(\gamma(t_0+\delta))}_{\to\, 0}$$

$$= \begin{cases} \varepsilon & \text{if } \gamma(t_1) \in \partial U \\ r(q) & \text{if } \gamma(t_1) \in U \end{cases}$$

$$\geq r(q) \quad \text{since } U = \{r < \varepsilon\}$$

$$= |\exp_p^{-1}(q)|$$

equality $\Rightarrow$ $t_0 = 0,\ t_1 = 1$ (so $\gamma[0,1] \subset U$)

$$\dot\theta^i \equiv 0,\quad \dot r > 0$$

$$\Rightarrow \gamma'(t) = \dot r\,\partial_r \quad \text{and} \quad |\gamma'| = \dot r^2 = \text{const}$$

$$= a\,\partial_r$$

$$\Rightarrow \gamma(t) = (r_0 + at,\ \theta^i_0)$$

$\gamma(t) \xrightarrow[t\to 0]{} (0, *)$ since $\gamma(0) = p$

so $r_0 = 0$

$$\gamma(1) = q = (r(q), \theta^i(q))$$

$$\Rightarrow a = r(q),\quad \theta^i_0 = \theta^i(q)$$

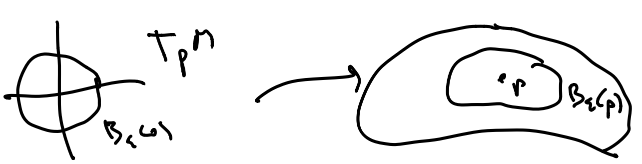
$\Rightarrow \gamma(t) = (r(q)t,\ \theta^i(q))$ in polar coords

$$= \exp_p(t\, r(q)\,\theta^i(q)) \qquad \square$$

Cor: ① in normal nbd @ p, $d(p,q) = r(q)$
$= |\exp_p^{-1}(q)|$

② $\exp_p B_\varepsilon(0) = B_\varepsilon(p) = \{q \in M : d(p,q) < \varepsilon\}$

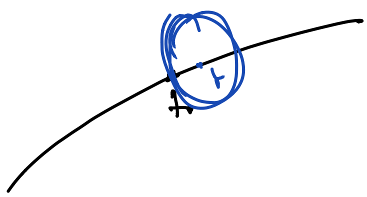
(provided $\exp_p|_{B_\varepsilon(0)}$ = diffeo)



thm: γ = pw. reg curve, const speed, then γ = geodesic $\Leftrightarrow$ $\underline{\gamma \text{ loc. minz}}$

i.e. $\forall t$, $d(\gamma(t), \gamma(s)) = L\gamma|_{[t,s]}$ for s near t

proof: ($\Rightarrow$) $\gamma : [a,b] \to M$, take $t_0 \in [a,b]$



$\exists\, \varepsilon > 0$ s.t. $\forall t$ s.t. $|t - t_0| < \varepsilon$

$\exp_{\gamma(t)} : B_{2\varepsilon}(0) \to M$ = diffeom

uniquess of geodesics $\Rightarrow$ $\gamma(t+s) = \exp_{\gamma(t)}(s\,\gamma'(t))$ $|s| < 2\varepsilon$

$\Rightarrow d(\gamma(t+s), \gamma(t)) = s\,|\gamma'(t)|$

$= L\gamma|_{(t, t+s)}$

$\Rightarrow d(\gamma(t), \gamma(t')) = L\gamma|_{[t,t']}$ for t, t' with ε of t_0

($\Leftarrow$) γ loc. minz, $t \in [a,b]$

$\hookrightarrow d(\gamma(t+s), \gamma(t)) = L\gamma|_{[t, t+s]}$ for $|s| < \varepsilon$

wlog $\exp_{\gamma(t)} : B_\varepsilon(0) \to M$ diffeom

$\Rightarrow \gamma(t+s) = \exp_{\gamma(t)}(sv)$ for $v = \gamma'(t) \neq 0$

$\Rightarrow \gamma$ geodesic $\square$

Rmk: "calibration argument"

(side) $\operatorname{grad} r \longrightarrow |\operatorname{grad} r| = 1$

$$\hookrightarrow L\gamma = \int |\gamma'| \geq \int \gamma' \cdot (\operatorname{grad} r)$$

$$= \int \frac{d}{dt} r(\gamma(t))$$

$$= r(\gamma(1)) - r(\gamma(0))$$

if $M^n, M'^n \subset N^{n+k}$ submanifolds, closed, $M - M' = \partial U$ homologous

if $\omega = n$-form on N st. $|\omega| \leq 1$, $d\omega = 0$,

"calibration for M" $\quad \underline{\omega|_M = dV_M}$,

$\Rightarrow \operatorname{vol}(M) \leq \operatorname{vol}(M')$

<u>proof:</u> $\operatorname{vol}(M') = \int_{M'} dV_{M'} \geq \int_{M'} \omega$

$$= \int_M \omega - \underbrace{\int_U d\omega}_{=0}$$

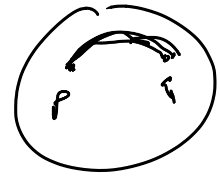
$$= \int_M dV_M = \operatorname{vol}(M) \qquad \square$$

<u>Convexity</u>

$U \subset (M, g)$ = (geodesically) convex if $\forall p, q \in U$

$\exists!$ min'z geodesic $\gamma : p \to q$ in U

<u>propn</u>: $\forall p \in M$, $\exists R > 0$ st $B_R(p)$ = convex

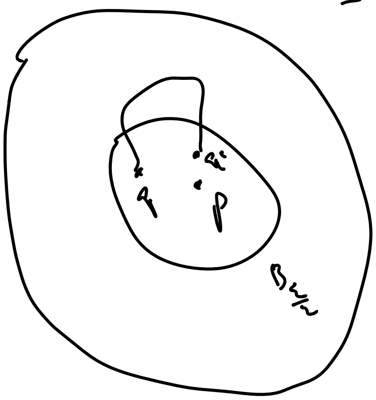


<u>proof.</u>

$\exists \varepsilon > 0$ s.t. $\forall q \in B_{\frac{\varepsilon}{2}}(p)$

$\exp_q$ = diffeo on $B_\varepsilon(0)$

(note $B_\varepsilon(q) \supset B_{\frac{\varepsilon}{2}}(p)$)



$\Rightarrow q, q' \in B_{\frac{\varepsilon}{2}}(p)$

$\exists$ minz geodesic $\gamma : q \longrightarrow q'$

WTS: γ lies inside $B_{\frac{\varepsilon}{2}}(p)$ (after shrinking ε)

define $f = \text{dist}(x,p)^2 = \sum_i {x^i}^2$ = smooth near p

in normal coords @ p

<u>claim</u>: $\nabla^2 f \geq 0$ in some ball $B_r(p)$, $r \leq \varepsilon$

recall $\text{Hess} f = \nabla^2 f$ = symmetric $(2,0)$-tensor

$$\nabla^2 f(X,Y) = X(Y(f)) - (\nabla_X Y) f$$

in normal coords, $\Gamma^k_{ij}(0) = 0$ and $x^i(0) = p$

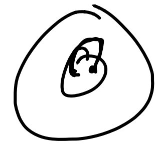
$$\Rightarrow \underbrace{\nabla^2 f(\partial_i, \partial_j)}_{\substack{\text{tensorial} \\ \downarrow \\ \text{ind of choice}}} = \partial_i \partial_j \left(\sum_p {x^p}^2 \right) \quad @\ 0$$

$$= 2\delta_{ij} = 2\langle \partial_i, \partial_j \rangle$$

$$> 0$$

of coords

$\Rightarrow \quad \nabla^2 f > 0 \quad$ in some $B_r(p)$



$q, q' \in B_{\frac{r}{2}}(p)$, $\quad \gamma$ = min'z geodesic $q \to q'$

by Δ-inequality, $\gamma \subset B_r(p)$

take $\gamma : [0, d] \to M$ param by arclength (PBAL)

$$d(p, \gamma(t)) \le d(p, q) + d(\gamma(t), q)$$

$$< \frac{r}{2} + \underline{\min(t, d-t)} \quad \text{(since could do } q' \text{ instead)}$$

$$\le r \quad \text{since } d \le r$$

now: $$\frac{d^2}{dt^2} f(\gamma(t)) = \gamma'(\gamma'(f))$$

$$= (\nabla^2 f)(\gamma', \gamma') \quad \text{since } \nabla_{\gamma'} \gamma' = 0$$

$$\geq 0$$

so $$\max f(\gamma(t)) = \max\left(f(\gamma(0)), f(\gamma(d)) \right)$$

$$= \max(f(q), f(q'))$$

$$< \left(\frac{r}{2}\right)^2$$

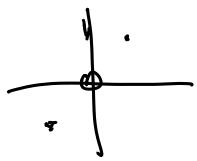
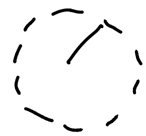
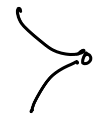
$\square$

[in general, if $\exp_p : B_r(0) \to M$ diffeo and $\operatorname{dist}(\cdot, p)^2$ convex in $B_r(p)$

$\Rightarrow B_{\frac{r}{2}}(p)$ is convex

Completeness

if $\bar{M}$ = mfd with ∂ eg open $\{|x|<1\}$

$\mathbb{R}^2 \setminus \{0\}$   

(M,g) = geodesically complete if every geodesic exists $\forall t \in \mathbb{R}$

Thm (Hopf-Rinow): (M,g) connected, $p \in M$, TFAE:

(a) $\exp_p$ defined $\forall v \in T_pM$

(b) closed, bdd sets are compact (Heine-Borel)

(c) (M,d) = complete metric space

(d) M = geodesically complete

(e) $\exists$ cpct sets $K_n \subset K_{n+1} \cdots$ $\cup K_n = M$, and $d(p, M \setminus K_n) \xrightarrow[n\to\infty]{} \infty$

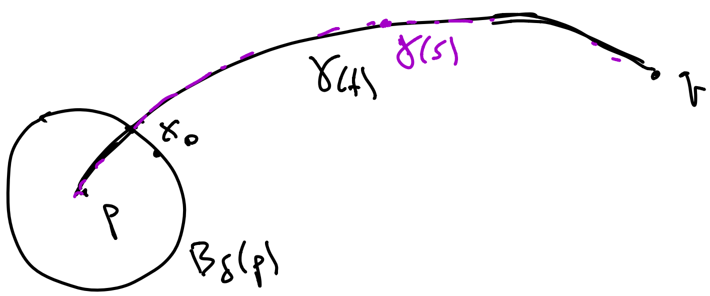
all of these imply:

(f) $\forall p,q$ $\exists$ minz geodesic $p \longrightarrow q$

Lemma: (M,g) connected, $p \in M$, $\exp_p$ defined on T_pM

$\Rightarrow \forall q \in M$, $\exists$ minz geodesic $p \to q$

proof of lemma: "choose a good direction"



choose $\delta > 0$ s.t. $\exp_p$ = diffeo ~ $B_{2\delta}(p)$

$\quad\hookrightarrow$ $\partial B_\delta(p)$ cpct

$\exists\, x_0 \in \partial B_\delta(p)$ closest to q

take γ = geodesic $p \longrightarrow x_0$, PBAL

$\quad\hookrightarrow$ extend to $t \in [0, \infty)$

set $r = d(p,q)$, WTS: $\gamma(r) = q$ $\left(\begin{array}{l} \text{since } |\gamma'| = 1 \\ \text{so } L\gamma|_{[0,r]} = r \end{array}\right)$

define $S = \{ s \in [0,r] \text{ s.t. } s + d(\gamma(s), q) = r \}$

$\quad\hookrightarrow$ ETS: $r \in S$

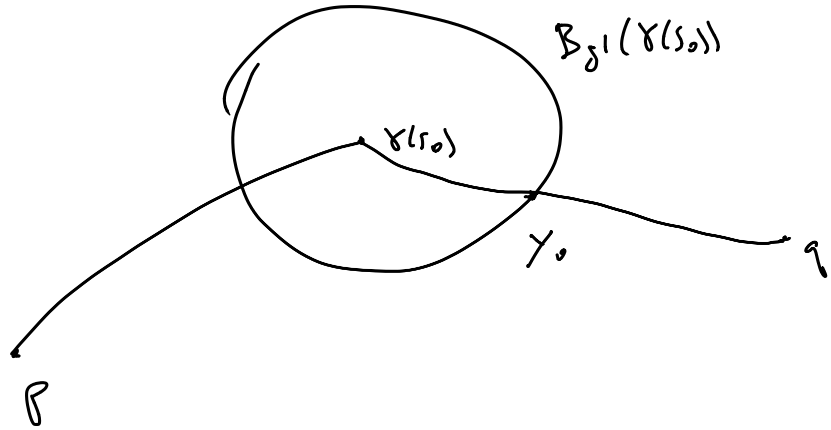
note S closed (since continuous in s)

$\quad 0 \in S$ $(\neq \emptyset)$

let $s_0 = \max S \leq r$

WTS: if $s_0 < r$ then $\exists\, \delta' > 0$ st. $s_0 + \delta' \in S$ $\lightning$

$\quad\hookrightarrow$ spse $s_0 < r$



choose δ' s.t. $\exp_{\gamma(s_0)}$ = diff on $B_{2\delta'}(0)$, $s_0 + 2\delta' < r$

↳ $\exists\, y_0 \in \partial B_{\delta'}(\gamma(s_0))$ closest to q

claim: $\gamma(s_0 + \delta') = y_0$

first, $d(\gamma(s_0), q) = \delta' + d(y_0, q)$

↑ if τ = pw. reg curve $\gamma(s_0) \longrightarrow q$

then $d(\tau(t), \gamma(s_0))$ continuous, $0 \longrightarrow r$

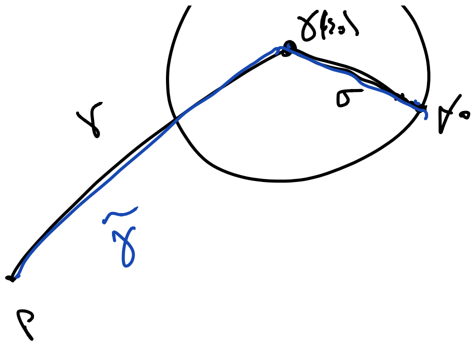
and $B_{2\delta'}(\gamma(s_0)) \not\ni q \Rightarrow r \geq \delta'$

$\Rightarrow \exists\, t$ st. $\tau(t) \in \partial B_{\delta'}(\gamma(s_0))$

$\Rightarrow L\tau \geq \delta' + d(y_0, q)$

then $d(y_0, q) = d(\gamma(s_0), q) - \delta'$

$= r - s_0 - \delta'$



$$\tilde{\gamma}(s) = \begin{cases} \gamma(s) & s \le s_0 \\ \sigma(s-s_0) & s_0 \le s \le s_0+\delta' \end{cases}$$

then $d(\tilde{\gamma}(s_0+\delta'), p) = d(y_0, p)$

$$\geq d(p,q) - d(y_0,q)$$

$$= s_0 + \delta'$$

$$= L\tilde{\gamma}\big|_{[0, s_0+\delta']}$$

$$\geq d(\tilde{\gamma}(s_0+\delta'), p)$$

so $\tilde{\gamma}$ = minz $\Rightarrow$ $\tilde{\gamma}$ = geodesic

$\Rightarrow \tilde{\gamma} = \gamma$ by uniqueness

so $\gamma(s_0+\delta') = y_0$ and $d(\gamma(s_0+\delta'), q) + s_0 + \delta' = r$

$\Rightarrow s_0 + \delta' \in S$ $\lightning$ $\square$

proof of Hopf-Rinow:

a $\Rightarrow$ f for p st. $\exp_p$ defined on T_pM

a $\Rightarrow$ b A bdd $\Rightarrow A \subset \overline{B_R(p)}$ for some $R > 0$

$\Rightarrow A \subset \exp_p(\overline{B_R(0)})$ = cpct

A also closed $\Rightarrow$ A cpct

<u>b $\Rightarrow$ c</u> $\{x_n\}$ Cauchy seq. in (M,d)

$\Rightarrow \{x_n\}$ bdd

$\Rightarrow \overline{\{x_n\}} =$ cpct $\Rightarrow x_n$ converges

<u>c $\Rightarrow$ d</u> γ = geodesic PBAL (= param by arc length)

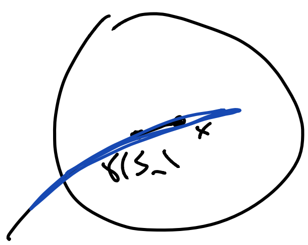
defined for $s < s_0$

take $s_n \nearrow s_0 \Rightarrow \{\gamma(s_n)\}_n =$ Cauchy

$$d(\gamma(s_n), \gamma(s_m)) \le L\gamma\big|_{[s_n, s_m]} = |s_n - s_m|$$

$\Rightarrow \gamma(s_n) \xrightarrow[n\to\infty]{} x$ (can define $\gamma(s_0) = x$)

$\left(\Rightarrow \gamma(s) \xrightarrow[s\to s_0]{} x\right)$



$\exists\, \delta > 0$ st. $\forall q \in B_\delta(x)$

$\exp_q$ diffeo^m on $B_\delta(0)$

so for s_n near s_0

$\Rightarrow \gamma(s) = \exp_{\gamma(s_n)}\left((s - s_n)\gamma'(s_n)\right)$ uniqueness of geodesics

but RHS exists $\forall\, |s - s_n| < \delta$

$\Rightarrow \gamma$ can be extended to $s < s_0 + \delta$

$\underline{d \Rightarrow a}$: trivial

$\underline{c \Leftrightarrow e}$: general metric space $\square$

Curvature

(M, g) Riem. mfd $\longrightarrow$ ∇ = L.C. connection

$\rightsquigarrow$ curvature tensor $\underline{R(X,Y)Z} = \underline{\nabla_X \nabla_Y} Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$= (3,1)$-tensor

$\rightsquigarrow$ Riemann curvature tensor $= Rm(X,Y,Z,W)$

$= R(X,Y,Z,W)$

$= \langle R(X,Y)Z, W \rangle$ $= (4,0)$-tensor

in $\underline{\text{coords}}$: $Rm_{ijkl} = R_{ijkl}$

$$= \langle \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k, \partial_l \rangle$$

$\underline{\text{Rmk}}$: Z vector field $= (0,1)$-tensor

∇Z $= (1,1)$-tensor

$\nabla^2 Z$ $= (2,1)$-tensor

$\hookrightarrow (\nabla^2 Z)(X,Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$

$\Rightarrow R(X,Y)Z = (\nabla^2 Z)(X,Y) - (\nabla^2 Z)(Y,X)$

" Rm = failure of $\nabla^2 Z$ to be symmetric

= obstruction to being loc. isometric to flat space "

....

Caution: Some authors define $R(X,Y)Z = -(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)$

for us, sectional curvature is "1221"

for others, — — — "1212"

check R is tensor: trivially $\mathbb{R}$-linear

need to check C^∞-linear

eg. Z slot: $R(X,Y)(fZ) = f R(X,Y)Z$

$$\text{LHS} = \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z)$$
$$- (f\nabla_{[X,Y]}Z + [X,Y]f\, Z)$$
$$= fR(X,Y)Z + X(f)\nabla_Y Z + X(Y(f))Z + Y(f)\nabla_X Z$$
$$- Y(f)\nabla_X Z - Y(X(f))Z - X(f)\nabla_Y Z$$
$$- [X,Y]f\, Z$$
$$= f R(X,Y)Z \qquad \square$$

Symmetries of R (0^{th} order):

① $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

"first Bianchi id"

② $R(X,Y,Z,W) = -R(Y,X,Z,W)$

③ $R(X,Y,Z,W) = -R(X,Y,W,Z)$

④ $R(X,Y,Z,W) = R(Z,W,X,Y)$

Proof: ② obvious

$R(X,Y,Z,W) = \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle$

① ETS: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

ETS: $R(\partial_i,\partial_j)\partial_k + R(\partial_j,\partial_k)\partial_i + R(\partial_k,\partial_i)\partial_j = 0$ since tensor

↳ $LHS = \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k + \boxed{\nabla_j\nabla_k\partial_i} - \boxed{\nabla_k\nabla_j\partial_i}$
$\quad + \nabla_k\nabla_i\partial_j - \boxed{\nabla_i\nabla_k\partial_j}$

$\boxed{\nabla_j\nabla_k\partial_i} = \nabla_j\nabla_i\partial_k$, $\quad \boxed{\nabla_k\nabla_j\partial_i} = \nabla_k\nabla_i\partial_j$, $\quad \boxed{\nabla_i\nabla_k\partial_j} = \nabla_i\nabla_j\partial_k$

since torsion free

$= 0$

③ ETS: $R(X,Y,Z,Z) = 0$

choose coords (s.t. $\Gamma^k_{ij}(p) = 0$, $g_{ij}(p) = \delta_{ij}$)

↳ ETS: $R(\partial_i,\partial_j,\partial_k,\partial_k) = 0$ (no summation convention)

(b/c tensor)

$LHS = \langle \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k, \partial_k\rangle$

$= \partial_i\langle\nabla_j\partial_k,\partial_k\rangle - \cancel{\langle\nabla_j\partial_k,\nabla_i\partial_k\rangle}$

$\quad - \partial_j\langle\nabla_i\partial_k,\partial_k\rangle + \cancel{\langle\nabla_i\partial_k,\nabla_j\partial_k\rangle}$

$= \partial_i\left(\partial_j\left(\frac{|\partial_k|^2}{2}\right)\right) - \partial_j\left(\partial_i\left(\frac{|\partial_k|^2}{2}\right)\right)$

$= 0$

④ sketch: (work in coords)

$R_{ijkl} + R_{jkil} + R_{kijl} = 0 \quad +$

$R_{jkli} + R_{klji} + \cdots = 0 \quad +$

R_{klij} . . . $= 0$ —

R_{lijk} . . . $= 0$ —

$2R_{ijkl} - 2R_{klij} = 0$ $\square$

<u>special case</u> : <u>$n=2$</u> : $\{e_1, e_2\}$ = ON basis for T_pM

↳ (up to sign) only non-zero entry of R

$= R(e_1, e_2, e_2, e_1)$

= Gauss curvature of M @ p

in general, if $\{v_1, v_2\}$ = basis for T_pM

↳ $R(v_1, v_2, v_2, v_1) = \underbrace{|v_1 \wedge v_2|^2}_{} R(e_1, e_2, e_2, e_1)$.

$= \det \langle v_i, v_j \rangle$ ←

$= |v_1|^2 |v_2|^2 - \langle v_1, v_2 \rangle^2$ ←

ie. Gauss curvature $K = \dfrac{R(v_1, v_2, v_2, v_1)}{|v_1 \wedge v_2|^2}$

<u>for general n</u> : if π = 2-d subspace of T_pM

$\{v_1, v_2\}$ = any basis for π

then (<u>defn</u>) sectional curvature of π

$= K(\pi) := \dfrac{R(v_1, v_2, v_2, v_1)}{|v_1 \wedge v_2|^2} = R(e_1, e_2, e_2, e_1)$

if $\{e_1, e_2\}$ = ON basis for T_pM

(interpretation: $K(\pi)$ = Gauss curvature of $\exp_p(\pi)$ @ p)

<u>Thm</u>: if $R, \overline{R}$ any two $(4,0)$-tensors satisfying the symmetries (1) – (3)
(i.e. the 0^{th} order symmetries of Rm)

and $R(X,Y,Y,X) = \overline{R}(X,Y,Y,X) \quad \forall X, Y$

$\Rightarrow R = \overline{R}$

<u>proof</u>: wlogspse $\overline{R} = 0 \Rightarrow$ WTS: $R = 0$

$$\hookrightarrow R(X+Y, Z, Z, X+Y) = 0$$
$$= R(X,Z,Z,Y) + R(Y,Z,Z,X)$$
$$= 2R(X,Z,Z,Y)$$

$$\Rightarrow R(X, Z+W, Z+W, Y) = 0$$
$$= R(X,Z,W,Y) + R(X,W,Z,Y)$$
$$\Rightarrow R(X,Z,W,Y) = -R(X,W,Z,Y) \leftarrow$$

$$\begin{array}{ccccc} 0 = R(X,Y,Z,W) & + & R(Y,Z,X,W) & + & R(Z,X,Y,W) \\ & & \shortparallel & & \shortparallel \\ & & -R(Y,X,Z,W) & & -R(X,Z,Y,W) \\ & & \shortparallel & & \shortparallel \\ & & R(X,Y,Z,W) & & R(X,Y,Z,W) \end{array}$$

$$\Rightarrow 0 = 3R(X,Y,Z,W)$$ $\square$

Cor: if $K_p(\pi) = k$ $\forall$ 2-planes π "isotropic"

$\Rightarrow R|_p(X,Y,Z,W) = k\left(\langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle \right)$

i.e. $R_{ijkl}|_p = k\left(g_{il} g_{jk} - g_{ik} g_{jl} \right)$

Proof: RHS satisfies same symmetries as R $\square$

= (4,0)-tensor

Ricci curvature $\quad Ric_p(X,Y) = \sum_{i=1}^{n} R|_p(X, e_i, e_i, Y)$ $\quad e_i$ = ON basis for T_pM

= symmetric (2,0)-tensor

in coords: $Ric_{ij} = R_{ik}{}^{k}{}_{j} = R_{kij}{}^{k} \ldots$

"trace R in middle 2 slots"

↳ $Ric(X,X)$ = average of sectional curvatures over planes containing X

Scalar curvature $\quad Scal_p = S(p) = \sum_{i=1}^{n} Ric_p(e_i, e_i)$

$= tr_g Ric$

$= Ric_k{}^{k} = g^{kp} Ric_{kp}$

Rmk: if $n=2$, then $S = Ric(e_1,e_1) + Ric(e_2,e_2)$

$= R(e_1,e_2,e_2,e_1) + R(e_2,e_1,e_1,e_2)$

$= 2K$

(and $R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$, $Ric_{ij} = K g_{ij}$)

<u>1st order symmetry</u>: second Bianchi identity

$$0 = (\nabla_T R)(X,Y,Z,W) + (\nabla_X R)(Y,T,Z,W) + (\nabla_Y R)(T,X,Z,W)$$

(HW...) in coords: $\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0$

<u>Ex</u>: (M,g) = <u>Einstein</u> if $Ric = \lambda(p)\, g$ for some function λ

["$Ric = \lambda g$" , stationary pts for $\int_M S\, dV_g$

$\hookrightarrow$ trace $(\ldots) \Rightarrow \operatorname{tr} Ric = S$

$$= \lambda \operatorname{tr}(g) = \lambda n$$

$$\Rightarrow Ric = \frac{1}{n} S g$$

<u>claim</u>: $n \geq 3$, S = locally const

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0$$

trace over j,k

$$\nabla_p \underbrace{R_{ik}{}^{k}{}_{l}} + \underbrace{\nabla_i R_{kp}{}^{k}{}_{l}}_{= -\nabla_i \underbrace{R_{pk}{}^{k}{}_{l}}} + \nabla_k R_{pi}{}^{k}{}_{l} = 0$$

$$\nabla_p Ric_{il} - \nabla_i Ric_{pl} - \nabla_k R_{pil}{}^{k} = 0$$

trace over i, ℓ

$$\nabla_p S - \nabla_i Ric_p{}^i - \nabla_k Ric_p{}^k = 0$$

$\boxed{\text{so}}$ $\nabla_i Ric_p{}^i = \frac{1}{2} \nabla_p S$

" $\operatorname{div} Ric = \frac{1}{2} \nabla S$ "

Einstein $\Rightarrow$ $\frac{1}{2} \nabla_p S = \nabla_i Ric_p{}^i = \nabla_i (g^{ik} Ric_{pk})$

$$= \nabla_i \left(\frac{1}{n} S \delta_{ip}\right) \qquad \nabla_i \left(\frac{1}{n} g^{ik} S g_{pk}\right)$$

$$= \frac{1}{n} \nabla_p S$$

so $\left(\frac{1}{n} - \frac{1}{2}\right) \nabla_p S = 0$

$n \neq 2 \Rightarrow \partial_p S = 0 \quad \forall p$

claim: if $n = 3 \Rightarrow K(\pi) = \text{const} = \frac{1}{6} S$

$Ric = \frac{1}{3} S g$

$R_{1221} + R_{1331} = \frac{1}{3} S$ (plug in e_1, e_1)

$R_{2112} + R_{2332} = \frac{1}{3} S$ etc

$R_{3113} + R_{3223} = \frac{1}{3} S$

$$\Rightarrow \begin{aligned} + R_{1221} + R_{1331} &= \tfrac{1}{3} S \\ + R_{1221} + R_{2332} &= \tfrac{1}{3} S \\ - R_{1331} + R_{2332} &= \tfrac{1}{3} S \\ \hline 2 R_{1221} &= \tfrac{1}{3} S \end{aligned}$$

$\Rightarrow K = \frac{1}{6} S = \text{const}$

Ex: space form M^n_k = simply-connected Riem. n-mfd with const sectional curv $= k$

$\hookrightarrow$ $M^n_1 = S^n$ = sphere

(thm) $M^n_0 = \mathbb{R}^n$ = euclidean space

$M^n_{-1} = \mathbb{H}^n$ = hyperbolic space

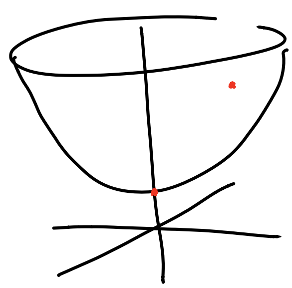
$\Rightarrow$ up to scaling, any (connected) M^n with K = const = quotient of S^n, $\mathbb{R}^n$, $\mathbb{H}^n$

Hilbert's thm $\Rightarrow$ cannot immerse $\mathbb{H}^2$ into $\mathbb{R}^3$

models for $\mathbb{H}^n$:

(1) $(\mathbb{R}^{n,1}, Q = -v_0 w_0 + v_1 w_1 + \dots + v_n w_n)$, $v = (v_0, v_1, \dots, v_n)$

$M^n = \{ Q(x,x) = -1 \} = \{ -x_0^2 + x_1^2 + \dots + x_n^2 = -1 \}$



$(M, Q|_M) = \mathbb{H}^n$

symmetries of $\mathbb{R}^{n,1} = O(n,1)$

= matrices A s.t. $A^T Q A = Q$

$$Q = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$\hookrightarrow$ $O(n) \subset O(n,1)$

$$A_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & \end{bmatrix} \in O(n,1)$$

$$\begin{pmatrix} & & 1 & \\ & 0 & & \ddots \end{pmatrix}$$

$\to$ $O(n,1)$ preserve M^n, acts transitively

@ $p = (1, 0, 0 \ldots, 0)$, $T_pM = \{0\} \times \mathbb{R}^n$

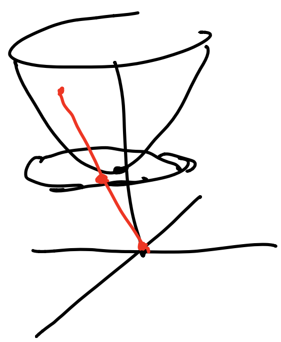
$\hookrightarrow$ given $v \in T_qM$, if $Aq = p$

$\Rightarrow Q(v,v) = Q(Av, Av) = g_{eucl}(Av, Av)$

≥ 0

② Poincare ball : $\left(B^n \subset \mathbb{R}^n,\ 4 \dfrac{\sum dx^{i2}}{(1-|x|^2)^2} \right)$

euclidean $|x|^2 = \sum x^{i2}$



③ half-space : $\left(\mathbb{R}^n_+ = \{x^n > 0\},\ \dfrac{\sum dx^{i2}}{(x^n)^2} \right)$

<u>Submanifolds</u>

$(\bar{M}^{n+k}, \bar{g})$, $M^n \subset \bar{M}$ <u>immersed</u> or <u>embedded</u> submfd

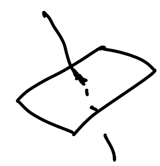
$\iota : M \to \bar{M}$ locally diffeo

near any $p \in M$ $\exists$ coords x^i on $\bar{M}$ near p s.t. $M = \{x^{n+1} = x^{n+2} = \ldots = x^{n+k} = 0\}$

$p \in M$

↳ $T_p\bar{M} = T_pM \oplus T_p^{\perp}M$ = N_pM = normal space to M

$\bar{g}$ - orthogonal decomp



$$X = X^{T} + X^{\perp}$$

↳ $\bar{g}$ induces Riem metric $g = \bar{g}\big|_{T_pM}$ on M

↳ LC connection $\bar{\nabla}$ of $\bar{g}$ induces LC. connection of g (lemma)

via $\nabla_X Y = (\bar{\nabla}_X Y)^{T}$

(X, Y vector fields in M → extend to fields in $\bar{M}$)

check ∇ torsion free, metric compatible

X, Y, Z tangential fields on M

↳ $X\langle Y, Z\rangle = \langle \bar{\nabla}_X Y, Z\rangle + \langle Y, \bar{\nabla}_X Z\rangle$

$= \langle (\bar{\nabla}_X Y)^{T}, Z\rangle + \langle Y, (\bar{\nabla}_X Z)^{T}\rangle$ ✓

x^i coords on M → $X = X^i \partial_i$, $Y = Y^i \partial_i$

$\Rightarrow [X, Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j)\partial_j$ = tangential

↳ $\nabla_X Y - \nabla_Y X = (\bar{\nabla}_X Y - \bar{\nabla}_Y X)^{T}$

$= [X, Y]^{T} = [X, Y]$ ✓

<u>second fundamental form</u>: $B(X, Y) = (\bar{\nabla}_X Y)^{\perp}$ $\quad X, Y \in \mathfrak{X}(M)$

$\hookrightarrow \bar{\nabla}_X Y = \nabla_X Y + B(X,Y)$ (Gauss eqn)

note: $B \in \Gamma(T^*M \otimes T^*M \otimes NM)$ (= tensor field)

$B(X,Y) = B(Y,X)$

$\hookrightarrow$ since $B(X,Y) - B(Y,X) = (\bar{\nabla}_X Y - \bar{\nabla}_Y X)^\perp$

$= [X,Y]^\perp = 0$

note: $\nu \in \Gamma(NM) \Rightarrow \langle B(X,Y), \nu \rangle$ = (2,0) - tensor on M

and $\underline{\langle B(X,Y), \nu \rangle = - \langle \bar{\nabla}_X \nu, Y \rangle}$ (Weingarten eqn)

$\hookrightarrow$ since $\langle \nu, Y \rangle = 0$

$\Rightarrow X \langle \nu, Y \rangle = 0 = \langle \bar{\nabla}_X \nu, Y \rangle + \underbrace{\langle \nu, \bar{\nabla}_X Y \rangle}$

$= \langle \nu, (\bar{\nabla}_X Y)^\perp \rangle$

Gauss equation: $\bar{R}(X,Y,Z,W) = R(X,Y,Z,W)$

$X, Y, Z, W \in \underline{T_p M}$ $\quad - \langle B(X,W), B(Y,Z) \rangle$

$+ \langle B(X,Z), B(Y,W) \rangle$

proof: extend X, Y, Z, W to fields on $\bar{M}$

s.t. on M they are tangential

$\Rightarrow \bar{R}(X,Y,Z,W) = \langle \underset{①}{\bar{\nabla}_X \bar{\nabla}_Y Z} - \underset{②}{\bar{\nabla}_Y \bar{\nabla}_X Z} - \underset{③}{\bar{\nabla}_{[X,Y]} Z}, W \rangle$

$$① = \langle \bar\nabla_X \bar\nabla_Y Z, W\rangle$$

$$= \langle \bar\nabla_X (\nabla_Y Z + B(Y,Z)), W\rangle$$

$$= \langle \underline{\nabla_X \nabla_Y Z} + \underbrace{\cancel{B(X, \nabla_Y Z)}}_{\in NM}{}^{\,0} + \underbrace{\bar\nabla_X B(Y,Z)}, W\rangle \quad \in TM$$

$$\langle B(Y,Z), W\rangle = 0$$

$$\Rightarrow X\langle B(Y,Z), W\rangle = 0 = \langle \bar\nabla_X B(Y,Z), W\rangle + \langle \underset{NM}{B(Y,Z)}, \underline{\bar\nabla_X W}\rangle$$

$$\Rightarrow \langle \bar\nabla_X B(Y,Z), W\rangle = -\langle B(Y,Z), B(X,W)\rangle$$

So $① = \langle \nabla_X \nabla_Y Z, W\rangle - \langle B(Y,Z), B(X,W)\rangle$

$$② = -\langle \nabla_Y \nabla_X Z, W\rangle + \langle B(X,Z), B(Y,W)\rangle$$

$$③ = -\langle \bar\nabla_{[X,Y]} Z, W\rangle$$

$$= -\langle (\bar\nabla_{[X,Y]} Z)^T, W\rangle = -\langle \nabla_{[X,Y]} Z, W\rangle$$

$$① + ② + ③ = R(X,Y,Z,W) - \langle B(Y,Z), B(X,W)\rangle + \langle B(X,Z), B(Y,W)\rangle$$

$\square$

<u>special case:</u> $M^2 \subset \mathbb{R}^3$ $\quad \{e_1, e_2\}$ = ON basis for T_pM

$$\hookrightarrow \underline{R(e_1,e_2,e_2,e_1)} = \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)|^2$$

if $\nu =$ (loc.) unit normal, $h(X,Y) = \langle B(X,Y), \nu\rangle$
= scalar second f.f.

$\Rightarrow R(e_1,e_2,e_2,e_1) = h(e_1,e_1)\,h(e_2,e_2) - h(e_1,e_2)^2$
$= h_{11}h_{22} - h_{12}^2$
$= \det(h)$ = Gauss curvature

Codazzi equation: $M^n \subset \bar{M}^{n+1}$, ν = loc. choice of unit normal

↳ $h(X,Y) = \langle B(X,Y), \nu\rangle$

$$\Rightarrow (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \bar{R}(X,Y,Z,\nu)$$

proof: $X,Y,Z \in T_pM$ → extend $X,Y,Z \in \mathfrak{X}(\bar{M})$
$p\in M$

s.t. $X|_M \in TM$ ditto for Y,Z
and $\nabla X|_p = 0$

[choose coords x^i in M near p, $x^i(0)=p$
↳ define $X = X(0) + a^i_j x^j \partial_i$ ....
→ choose coords $\bar{x}^i$ in $\bar{M}$ st. $M = \{\bar{x}^{n+1}=0\}$
↳ define $X(\bar{x}^1,\ldots,\bar{x}^{n+1}) = X(\bar{x}^1,\ldots,\bar{x}^n,0)$]

$$\bar{R}(X,Y,Z,\nu) = \langle \underset{①}{\bar{\nabla}_X\bar{\nabla}_Y Z} - \underset{②}{\bar{\nabla}_Y\bar{\nabla}_X Z} - \underset{③}{\bar{\nabla}_{[X,Y]}Z}, \nu\rangle$$

$$① = \langle \bar\nabla_X \bar\nabla_Y Z, \nu \rangle$$

since $h = \langle B, \nu \rangle$ $= (2,0)$-tensor

$$= \langle \bar\nabla_X (\nabla_Y Z + h(Y,Z)\nu), \nu \rangle$$

$$= \langle \underbrace{\cancel{\nabla_X \nabla_Y Z}}_{0 \text{ since tangential}} + \underbrace{\cancel{h(X,\nabla_Y Z)\nu}}_{\to 0 \text{ @ } p} + X(h(Y,Z))\nu + \underbrace{h(Y,Z)\bar\nabla_X \nu}_{\to 0}, \nu \rangle$$

$\langle \nu, \nu \rangle = 1$

$X(\ldots) \Rightarrow 0 = 2\langle \bar\nabla_X \nu, \nu \rangle$

$$= X(h(Y,Z))$$

$$= \underline{(\nabla_X h)(Y,Z)} + \underbrace{\cancel{h(\nabla_X Y, Z)}}_{\to 0 \text{ @ } p} + \underbrace{\cancel{h(Y, \nabla_X Z)}}_{\to 0 \text{ @ } p}$$

$$② = -\langle \bar\nabla_Y \bar\nabla_X Z, \nu \rangle = -(\nabla_Y h)(X,Z)$$

$$③ = -\langle \bar\nabla_{[X,Y]} Z, \nu \rangle = -\langle \underline{(\bar\nabla_{[X,Y]} Z)^\perp}, \nu \rangle$$

$$= -h([X,Y], Z)$$

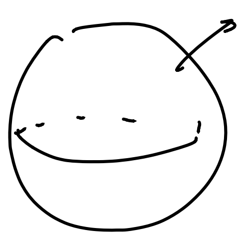
$$= -h(\nabla_X Y - \nabla_Y X, Z)$$

$$= 0 \quad @ \, p \qquad \square$$

Rmk: if $\bar M = \mathbb{R}^{n+1}$: $\nabla_\cdot h(\cdot,\cdot)$ tot. symmetric

Ex: $S^n \subset \mathbb{R}^{n+1}$

unit sphere



$\vec\nu = \vec x$

$D_X Y$ = directional derivative in $\mathbb{R}^{n+1}$

$h(X,Y) = \langle B(X,Y), \nu \rangle$

$= \langle D_X Y, \nu \rangle$

$= X\cancel{\langle Y, \nu \rangle}^{0} - \langle Y, D_X \nu \rangle$

$= -\langle Y, D_X \vec{x} \rangle$ so $h_{ij} = -g_{ij}$

$= -\langle Y, X \rangle$

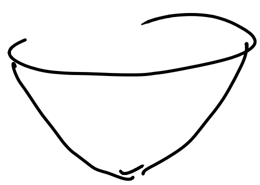
Gauss eqn: $R(X,Y,Z,W) = h(X,W)h(Y,Z) - h(X,Z)h(Y,W)$

$= \langle X,W \rangle \langle Y,Z \rangle - \langle X,Z \rangle \langle Y,W \rangle$

$\Rightarrow$ const sectional curvature $=1$

Ex $\mathbb{H}^n \subset \mathbb{R}^{n,1} = \{ \mathbb{R}^{n+1},\ Q(x,y) = x^T Q y = x^T \boxed{\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}} y \}$

$\mathbb{H}^n = \{ Q(x,x) = -1 \}_{x_0 > 0}$, $g_{\mathbb{H}^n} = Q|_{\mathbb{H}^n}$



if $x(t)$ path in $\mathbb{H}^n$

$\Rightarrow Q(x, \dot{x}) = 0$ and $Q|_{T_xM}$ = Riem. metric

$\Rightarrow x \notin T_xM$ and $Q(x,v) = 0 \ \forall v \in T_xM$

same argument as before: $h(X,Y) = Q(B(X,Y), \nu)$

$h(X,Y) = -Q(X,Y)$

$\hookrightarrow R(X,Y,Z,W) = Q(B(X,W), B(Y,Z)) - Q(B(X,Z), B(Y,W))$

$= h(X,W)h(Y,Z)\underbrace{Q(\vec{x},\vec{x})}_{=-1} - h(X,Z)h(Y,Z)\underbrace{Q(\vec{x},\vec{x})}_{=-1}$

$$= - \left(\langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, Z\rangle \right)$$

Const Sect curv. $= -1$

Ex: $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n+k}$ immersion

$\to \dfrac{\partial F}{\partial x^i}$ = coordinate fields on $F(U)$

$$\Rightarrow g_{ij} = \frac{\partial F}{\partial x^i} \cdot \frac{\partial F}{\partial x^j}$$

$$\Rightarrow B_{ij} = \left(\frac{\partial^2 F}{\partial x^i \partial x^j} \right)^{\perp}$$

Ex: $\{ x^{n+1} = f(x^1, \dots, x^n) \} \to F(x^1, \dots, x^n) = (x^1, \dots, x^n, f(x^1, \dots, x^n))$

$\hookrightarrow \partial_i F = (e_i, \partial_i f) \,, \quad \partial^2_{ij} F = (0, \partial^2_{ij} f)$

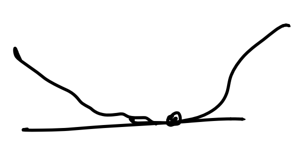
$\hookrightarrow g_{ij} = (e_i, \partial_i f) \cdot (e_j, \partial_j f) = \delta_{ij} + \partial_i f \, \partial_j f$

$$\to \sqrt{\det g_{ij}} = \sqrt{1 + |Df|^2}$$

$\hookrightarrow$ upwards normal $\nu = \dfrac{(-\partial_1 f, -\partial_2 f, \dots, -\partial_n f, 1)}{\sqrt{1 + |Df|^2}}$

$$h_{ij} = \langle B(\partial_i, \partial_j), \nu \rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \nu \right\rangle = \frac{\partial^2_{ij} f}{\sqrt{1 + |Df|^2}}$$



(if $Df = 0$ @ p $\Rightarrow g_{ij} = \delta_{ij}$, $h_{ij} = \partial^2_{ij} f$ @ p)

Gauss/Codazzi for $M^2 \subset \mathbb{R}^3$

week of March 22: midterm

$F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ immersion

$\partial_i F$ = tangent vectors

$\partial_i \partial_j F \left(= D_{\partial_i} \partial_j \right) = (\partial_i \partial_j F)^T + (\partial_i \partial_j F)^\perp$

$= \nabla_{\partial_i} \partial_j + h(\partial_i, \partial_j)\nu$

some choice of unit normal

↳ second partials commute $\Leftrightarrow$ ∇ torsion free, h symmetric

$\partial_k \partial_j \partial_i F = D_{\partial_k} D_{\partial_j} \partial_i$

$= D_{\partial_k} \left(\nabla_{\partial_j} \partial_i + h_{ij} \nu \right)$

$= \nabla_k \nabla_j \partial_i + h(\partial_k, \nabla_j \partial_i)\nu + \partial_k h_{ij} \nu - h_{ij} h_k{}^p \partial_p$

↳ third partials commute

$\Longleftrightarrow$ Gauss eqn $R_{ijkl} = h_{il} h_{jk} - h_{ik} h_{jl}$

Codazzi $\nabla_i h_{jk} = \nabla_j h_{ik}$

aside: $h_{ij} = \langle \partial_i \partial_j F, \nu \rangle$ ν unit normal

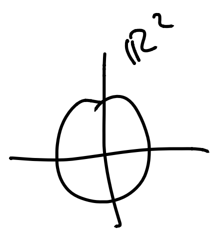
$\Rightarrow \langle \partial_i \nu, \partial_j \rangle = -h_{ij}$ and $\langle \partial_i \nu, \nu \rangle = 0$

so $\partial_i \nu = a_i^j \partial_j$ and $-h_{ij} = \langle \partial_i \nu, \partial_j \rangle = a_i^p g_{pj}$

$$\Rightarrow \partial_i \nu = -h_{ip} g^{pk} \partial_k$$

Q: do g_{ij}, h_{ij} uniquely determine F?

given smooth families of matrices $g_{ij}, h_{ij} : B_1 \subset \mathbb{R}^2 \to \mathbb{R}^4$



g_{ij}, h_{ij}

$\overset{?}{\Rightarrow} \exists ?$ immersion $F : B_1 \to \mathbb{R}^3$

s.t. $g_{ij} = \partial_i F \cdot \partial_j F$

$h_{ij} = \partial_i \partial_j F \cdot \nu$

for unit normal ν

Thm (Bonnet): provided g_{ij} = symmetric, positive def

h_{ij} = symmetric

and g_{ij}, h_{ij} satisfy Gauss, Codazzi eqns

$\Rightarrow$ yes, $\exists$ immersion $F : B_1^2 \to \mathbb{R}^3$ s.t.

F unique up to rigid motion

heuristic: $g_{ij} = \partial_i F \cdot \partial_j F$ 3 choices $\qquad F$ 3 eqns

$h_{ij} = \partial_i \partial_j F \cdot \nu$ 3 choices

Gauss eqn -1 restriction

Codazzi $\nabla h_{..}$ -2 restrictions

---

3 choices

proof: if $F : B_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ immersion w/ unit normal N

s.t. $g_{ij} = \partial_i F \cdot \partial_j F \quad h_{ij} = \partial^2_{ij} F \cdot N$

then $E_i = \partial_i F$ solve:
and N

$$(*) \begin{cases} \underline{\partial_i E_j = \Gamma_{ij}^k E_k + h_{ij} N} \\ \underline{\underline{\partial_i N = -h_i^{\ k} E_k = -h_{ip} g^{pk} E_k}} \end{cases}$$

for $\Gamma_{ij}^k = \frac{1}{2} g^{kp} (\partial_i g_{pj} + \partial_j g_{pi} - \partial_p g_{ij})$

idea: find $E_1, E_2, E_3 = N : B_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ solving $(*)$

(with some initial condition @ $(0,0)$)

$\longrightarrow$ integrate to get F

rewrite $(*)$ as

$$(**) \begin{cases} \partial_i E_\alpha = A_{i\alpha}^{\ \ \beta} E_\beta & i = 1,2 \quad \alpha = 1,2,3 \\ \text{i.c.} \quad E_1(0,0) = (a,0,0) & a > 0, \ c > 0 \\ \quad\quad E_2(0,0) = (b,c,0) & g_{ij}\big|_0 = E_i \cdot E_j \big|_0 \\ \quad\quad N(0,0) = (0,0,1) & \end{cases}$$

(So, asking for $N = (0,0,1)$
$\partial_1 F$ pts in positive x^1-dir
$\partial_2 F$ pts in positive x^2-dir)

note: Gauss/Codazzi eqns

$$\Longleftrightarrow \partial_1 A_{2\alpha}^{\ \ \beta} \delta_{\beta\gamma} + A_{2\alpha}^{\ \ \beta} A_{1\beta}^{\ \ \gamma}$$
$$= \partial_2 A_{1\alpha}^{\ \ \beta} \delta_{\beta\gamma} + A_{1\alpha}^{\ \ \beta} A_{2\beta}^{\ \ \gamma}$$

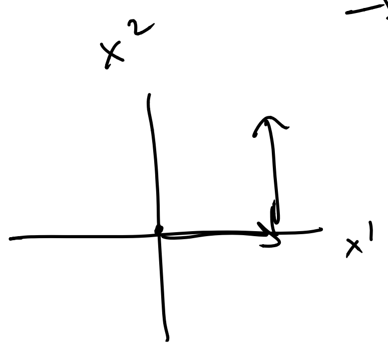
rmk: if $\gamma : [0,1] \to B_1$ regular path

then E_α solving (**) along γ

$$\iff \begin{cases} \frac{d}{dt} E_\alpha(\gamma(t)) = \dot\gamma^i A_{i\alpha}^{\beta} E_\beta \\ E_\alpha(\gamma(0)) = \text{initial cond} \end{cases}$$

= first order linear ODE

$\Rightarrow$ can solve $\forall\, t \in [0,1]$



first find $E_\alpha(x^1, 0)$ by solving ODE $\begin{cases} \partial_1 E_\alpha = A_{1\alpha}^{\beta} E_\beta \\ E_\alpha(0,0) = \text{prescribed} \end{cases}$

find $E_\alpha(x^1, x^2)$ by solving $\begin{cases} \partial_2 E_\alpha = A_{2\alpha}^{\beta} E_\beta \\ E_\alpha(x^1, 0) = E_\alpha(x^1, 0) \end{cases}$

$\Rightarrow \partial_2 E_\alpha = A_{2\alpha}^{\beta} E_\beta$ on $(x^1, x^2) \in B_1$

claim: $\partial_1 E_\alpha = A_{1\alpha}^{\beta} E_\beta$ on B_1

define $T_\alpha = \partial_1 E_\alpha - A_{1\alpha}^{\beta} E_\beta$ ($= 0$ when $x^2 = 0$)

$\hookrightarrow \partial_2 T_\alpha = \partial_2 \partial_1 E_\alpha - \partial_2 (A_{1\alpha}^{\beta} E_\beta)$

$= \partial_1 (A_{2\alpha}^{\beta} E_\beta) - \partial_2 A_{1\alpha}^{\beta} E_\beta - A_{1\alpha}^{\beta} A_{2\beta}^{\gamma} E_\gamma$

b/c $\longrightarrow$

$$= \partial_1 (A_{2\alpha}{}^{\beta} E_\beta) - \partial_1 A_{2\alpha}{}^{\beta} E_\beta - A_{2\alpha}{}^{\beta} A_{1\beta}{}^{\gamma} E_\gamma$$

$$= A_{2\alpha}{}^{\beta} (\partial_1 E_\beta - A_{1\beta}{}^{\gamma} E_\gamma)$$

$$= A_{2\alpha}{}^{\beta} T_\beta$$

if $f = |T_1|^2 + |T_2|^2 + |T_3|^2$

$$\Rightarrow \partial_2 f \le C f \Rightarrow f(x^1, x^2) \le f(x^1, 0)\, e^{C|x^2|} = 0 \quad \checkmark$$

$\rightsquigarrow$ get $E_1, E_2, N^{\,/\!/ E_3}$ solving ⊛ with initial conditions ⊛⊛

claim: $g_{ij} = E_i \cdot E_j$, $|N|^2 = 1$, $N \perp E_1, E_2$

if $f_{ij} = g_{ij} - E_i \cdot E_j \longleftarrow$ use ⊛

$$\Rightarrow \partial_k f_{ij} = \cdots = -\Gamma_{ki}^{p} f_{pj} - \Gamma_{kj}^{p} f_{pi} + h_{ki} N\cdot E_j + h_{kj} N \cdot E_i$$

and $\partial_k N \cdot E_i = \cdots = -h_k{}^{p} f_{pi} + h_{ik}(N\cdot N - 1) + \Gamma_{ki}^{p} N \cdot E_p$

and $\partial_k N \cdot N = -2 h_k{}^{p} N \cdot E_p$

so if $F = \sum_{i,j} f_{ij}^2 + (N\cdot N - 1)^2 + (N\cdot E_1)^2 + (N \cdot E_2)^2$

$$\Rightarrow \partial_k F \le C F \Rightarrow F(p) \le F(0)\, e^{C|p|} = 0 \quad \checkmark$$

hence $\partial_k N \cdot E_i = -h_{kp}\, g^{pj} E_j$

since g_{ij}, h_{ij}, Γ_{ij}^{k} sym $\Rightarrow$ $\partial_1 E_2 = \partial_2 E_1$

$\Rightarrow$ define $F(x^1, x^2) = \int_\gamma E_1 \, dx^1 + E_2 \, dx^2$

for any path $\gamma : [0,1] \to (x^1, x^2)$

$\Rightarrow$ ind. of path (Green's thm)

and $\partial_i F = E_i$ $\square$

Gauss-Bonnet

Q: curvature $\leftrightarrow$ topology?

"local" "global"

Gauss-Bonnet: (M^2, g) compact, no boundary, oriented

$$\Rightarrow \int_M K \, dA = 2\pi \chi(M)$$

Gauss curvature

Euler characteristic

$= " \underline{V} - \underline{E} + \underline{F} "$

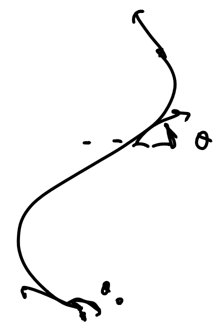
Ex: S^2 has $\chi = 2$

(1 pt) $-$ (1 curve) $+$ (2 faces) $= 2$

$\Rightarrow \int_{S^2} K \, dA = 4\pi$ for any Riem metric on sphere etc...

baby Gauss-Bonnet:

$\gamma : [a,b] \to \mathbb{R}^2$ regular curve PBAL



$\hookrightarrow \gamma' : [a,b] \to S^1$

$\Rightarrow \exists$ lift $\theta : [a,b] \to \mathbb{R}$

s.t. $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$

(unique up to multiples of 2π)

$\Rightarrow$ rotation angle of γ = $\mathrm{rot}(\gamma) := \theta(b) - \theta(a)$

" = net angle change by γ' "

rotation angle thm (Hopf): $\gamma : [a,b] \to \mathbb{R}^2$ = simple, closed, regular, positively oriented

$\Rightarrow \mathrm{rot}(\gamma) = 2\pi$

γ = simple, closed if $\gamma(a) = \gamma(b)$

$\gamma|_{[a,b)}$ injective

(no self-intersections)

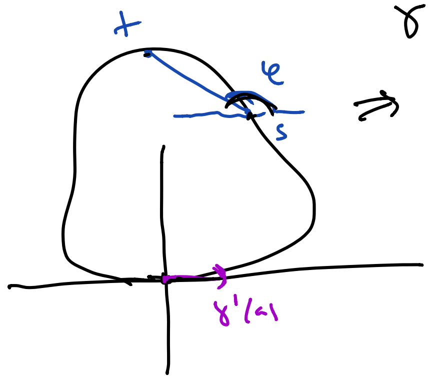
Fact: γ bounds a disk

$\hookrightarrow$ γ positively oriented if disk is to the left as you traverse curve

$\llcorner$ γ = simple, closed, regular if $\gamma' \neq 0$ at $\gamma'(a) = \gamma'(b)$

Proof: $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ reg. closed, simple PBAL

$\gamma(a) = \gamma(b)$, $\gamma'(a) = \gamma'(b)$



$\Rightarrow$ extends to C^1 curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$

s.t. $\gamma(t) = \gamma(t + (b-a))$ ..

wlog $\gamma(a) = 0$, $\gamma'(a) = (1,0)$

and $\gamma(t)$ lives in upper half plane

define $V : \{a \leq s \leq t \leq b\} \longrightarrow S^1$

$$(s,t) \longmapsto \begin{cases} \dfrac{\gamma(t) - \gamma(s)}{|\gamma(t) - \gamma(s)|} & \text{if } s \neq t,\ (s,t) \neq (a,b) \\ \gamma'(t) & \text{if } s = t \\ -\gamma'(a) & \text{if } (s,t) = (a,b) \end{cases}$$

claim: V continuous

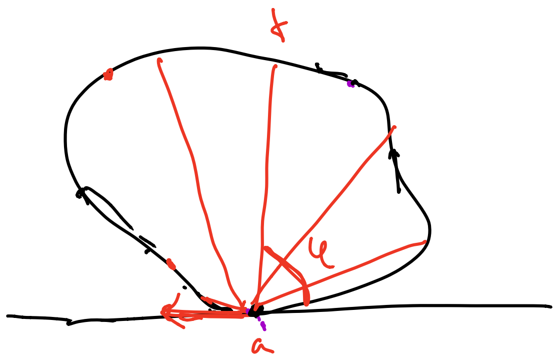
if $s \nearrow t$ then $\dfrac{\gamma(t) - \gamma(s)}{|\gamma(t) - \gamma(s)|} = \dfrac{\gamma(t) - \gamma(s)}{t-s} \, \dfrac{|t-s|}{|\gamma(t) - \gamma(s)|}$

$$\xrightarrow[s \nearrow t]{} \frac{\gamma'(t)}{|\gamma'(t)|} = \gamma'(t)$$

$\Rightarrow \exists$ lift $\varphi(s,t) : \{a \leq s \leq t \leq b\} \longrightarrow \mathbb{R}$

s.t. $V(s,t) = (\cos\varphi, \sin\varphi)$

wlog $\theta(a) = 0$, $\varphi(t,t) = \theta(t)$



$\varphi(a,a) = 0$

$\varphi(a,t) \in [0,\pi)$

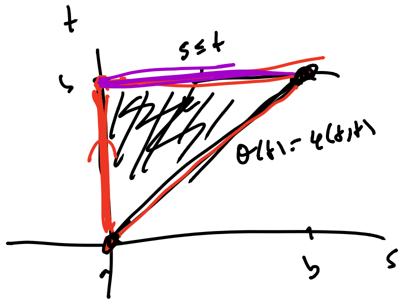
and $\varphi(a,b) = \pi + 2\pi N \quad N \in \mathbb{Z}$

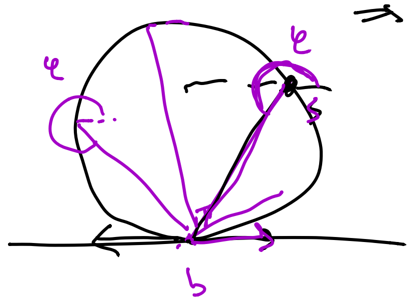
$\Rightarrow \varphi(a,b) = \pi$

$\varphi(s,b) \in [\pi, 2\pi]$

and $\varphi(b,b) = 2\pi + 2\pi N \quad N \in \mathbb{Z}$

$\Rightarrow \varphi(b,b) = 2\pi$

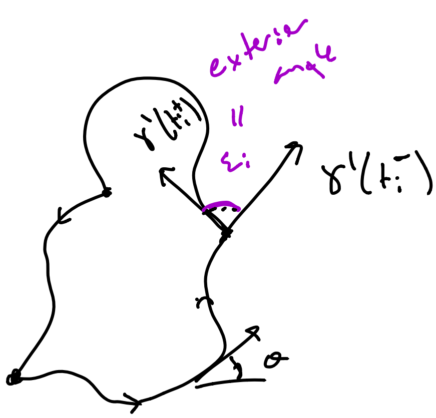




$\boxed{\text{So}}$ $\theta(b) - \theta(a) = \varphi(b,b) - \varphi(a,a) = 2\pi$ $\square$

$\gamma : [a,b] \to \mathbb{R}^2$ <u>curved polygon</u> if

γ = p.w. regular, $\gamma(a) = \gamma(b)$, $\gamma|_{[a,b)}$ injective

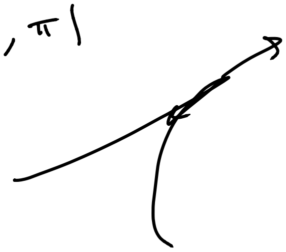


$\exists\ a = t_0 < t_1 < \dots < t_N = b$

st $\gamma|_{[t_i, t_{i+1}]}$ = regular

<u>ask that</u> $\varepsilon_i \in (-\pi, \pi]$

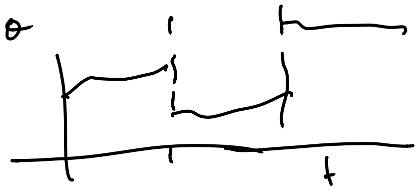
i.e. no cusp



extend γ to $\mathbb{R} \to \mathbb{R}$ periodic define $\theta : \mathbb{R} \to \mathbb{R}$ piecewise continuous

st $\gamma'(t) = (\cos\theta, \sin\theta)$ $\quad t \in [t_i, t_{i+1})$

and $\underbrace{\theta(t_i^+) - \theta(t_i^-)}$ $= \varepsilon_i$ = ext. angle $\in (-\pi, \pi)$



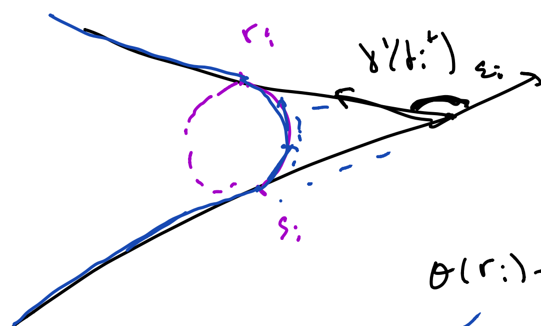
(unique up to choice of $\theta(a)$)

rotation $\operatorname{rot}\gamma = \theta(b) - \theta(a)$ $\left(= \sum_i \int_{t_i}^{t_{i+1}} \frac{d\theta}{ds}\, ds + \sum_i \varepsilon_i \right)$

<u>rotation angle thm 2</u> : γ = curved polygon, positively oriented

$\Rightarrow \operatorname{rot}(\gamma) = 2\pi = \int_{\partial\Omega} \frac{d\theta}{ds}\, ds + \sum \varepsilon_i$

<u>proof:</u>



$\theta(r_i) - \theta(s_i) \in (-\pi, \pi)$

replace γ with $\tilde{\gamma}$

$\Rightarrow \operatorname{rot}\gamma = \operatorname{rot}\tilde{\gamma} = 2\pi$ $\square$

<u>general metric</u> $g = g_{ij} = \langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$, e_i = std basis

$e_1 = (1, 0)$

$e_2 = (0, 1)$

$\hookrightarrow$ g-angle of $v, w = \dfrac{\langle v, w \rangle}{|v|_g |w|_g}$

$$E_1 = \frac{e_1}{|e_1|_g}, \quad E_2 = \frac{e_2 - \langle e_2, E_1\rangle E_1}{|\dots|_g}$$

$\Rightarrow$ E_1, E_2 is g-ON, E_1 = positive multiple of e_1, positively oriented $\blacksquare$

if $\gamma : [a,b] \to \mathbb{R}^2$ reg curve

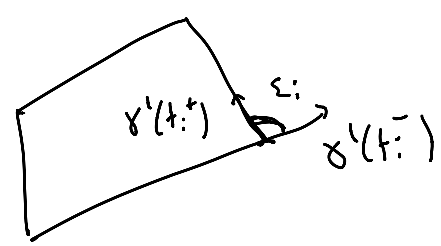
$\hookrightarrow \exists\ \theta_g : [a,b] \to \mathbb{R}$ s.t. $\frac{\gamma'}{|\gamma'|_g} = \cos\theta_g E_1 + \sin\theta_g E_2$

$\Rightarrow \operatorname{rot}_g(\gamma) = \theta_g(b) - \theta_g(a)$

= net g-angle traversed by γ'

$\gamma : [a,b] \to \mathbb{R}^2$ curved polygon

$\hookrightarrow$ exterior g-angle $\varepsilon_i \in (-\pi, \pi)$



$\left(\begin{array}{c} a = t_0 < t_1 < \dots < t_k = b \\ \text{st } \gamma|_{[t_i, t_{i+1}]} \text{ reg} \end{array} \right)$

$\hookrightarrow$ can define $\theta_g : [a,b] \longrightarrow \mathbb{R}$

s.t $\frac{\gamma'}{|\gamma'|_g} = \cos\theta E_1 + \sin\theta E_2$ for $t \neq t_i$

$\theta_g(t_i^+) - \theta_g(t_i^-) = \varepsilon_i = g$-ext. angle

<u>note</u>: θ_g and $\operatorname{rot}_g(\gamma) = \theta_g(b) - \theta_g(a)$

depend continuously on g

<u>rotation angle thm 3</u> : γ = curved polygon in $(\mathbb{R}^2, g)$ positively oriented

$$\Rightarrow \underline{\mathrm{rot}_g \gamma} = 2\pi = \int_\gamma \frac{d\theta_g}{ds}\, ds + \sum_i \varepsilon_i$$

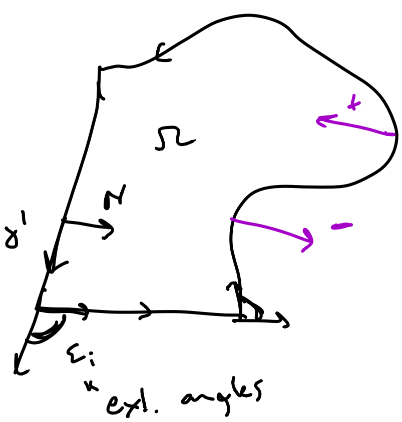
<u>proof</u>: Since $\gamma'(a^+) = \gamma'(b^+)$

$\Rightarrow \frac{1}{2\pi} \mathrm{rot}_g \gamma$ = integer

and continuous in g

$\Rightarrow g(s) = (1-s)\, g_{eucl} + s\, g \qquad s \in [0,1]$

and $\mathrm{rot}_{g(0)} \gamma = 2\pi = \mathrm{rot}_{g(1)} \gamma$ $\qquad \square$



$\Omega \subset (\mathbb{R}^2, g)$

$\partial\Omega$ curved polygon

$\hookrightarrow \gamma(\underline{s}) : [a,b] \to \mathbb{R}^2$ params $\partial\Omega$ by arclength

positively oriented

N = inward unit normal

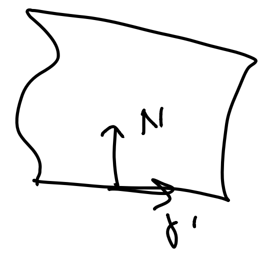
$\frac{D\gamma'}{ds}$ = <u>curvature vector</u> of γ

$\langle \frac{D\gamma'}{ds}, N \rangle$ = curvature scalar (circle has positive curvature) local Gauss-Bonnet: $\int_{\Omega} K\,dA + \int_{\partial\Omega} k_N\,ds + \sum_i \varepsilon_i = 2\pi$

Gauss curvature — curvature scalar — exterior angles

proof: E_1, E_2, θ_γ as before

↳ (rot angle thm) $\int_{\partial\Omega} \frac{d\theta}{ds}\,ds + \sum_i \varepsilon_i = 2\pi$



$$\gamma' = \cos\theta\, E_1 + \sin\theta\, E_2$$

↳ $N = J\gamma'$ $\quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= -\sin\theta\, E_1 + \cos\theta\, E_2$$

$$\Rightarrow \frac{D\gamma'}{ds} = -\sin\theta\,\theta' E_1 + \cos\theta\,\theta' E_2 + \cos\theta\,\nabla_{\gamma'}E_1 + \sin\theta\,\nabla_{\gamma'}E_2$$

$$= \theta' N + \cos\theta\,\nabla_{\gamma'}E_1 + \sin\theta\,\nabla_{\gamma'}E_2$$

$$\Rightarrow k_N = \left\langle \frac{D\gamma'}{ds}, N \right\rangle = \theta' + \cos\theta \langle N, \nabla_{\gamma'}E_1\rangle + \sin\theta\langle N, \nabla_{\gamma'}E_2\rangle$$

since E_i g-ON $\Rightarrow 1 = |E_i|_g^2, \quad 0 = \langle E_1, E_2\rangle$

$$\Rightarrow \langle \nabla_X E_i, E_i\rangle = 0, \quad \langle \nabla_X E_1, E_2\rangle = -\langle E_1, \nabla_X E_2\rangle$$

$\hookrightarrow$ define $\omega(X) = \langle E_1, \nabla_X E_2 \rangle \equiv -\langle \nabla_X E_1, E_2 \rangle$

then $\nabla_X E_1 = \underline{-\omega(X)} E_2$, $\nabla_X E_2 = \underline{\omega}(X) E_1$

so $\quad k_N = \theta' + \cos\theta \langle -\sin\theta \cancel{E_1} + \cos\theta E_2, -\omega(\gamma') E_2 \rangle$

$\qquad + \sin\theta \langle -\sin\theta E_1 + \cancel{\cos\theta E_2}, \omega(\gamma') E_1 \rangle$

$= \theta' - \omega(\gamma')$

$$\Rightarrow 2\pi = \sum_i \varepsilon_i + \int_{\partial\Omega} \frac{d\theta}{ds}\, ds$$

$$= \sum_i \varepsilon_i + \int_{\partial\Omega} k_N\, ds + \int_{\partial\Omega} \omega \quad '' \int_{\partial\Omega} \omega(\gamma')\, ds$$

Ω = domain with corners

$$= \sum_i \varepsilon_i + \int_{\partial\Omega} k_N\, ds + \underline{\int_{\Omega} d\omega}$$

<u>NTS</u>: $d\omega = K\, dA$

$$d\omega(E_1, E_2) = E_1 \omega(E_2) - E_2 \omega(E_1) - \omega[E_1, E_2]$$

$$= E_1 \langle \nabla_{E_2} E_2, E_1 \rangle - E_2 \langle \nabla_{E_1} E_2, E_1 \rangle - \langle \nabla_{[E_1, E_2]} E_2, E_1 \rangle$$

$$= \langle \nabla_{E_1} \nabla_{E_2} E_2, E_1 \rangle - \langle \nabla_{E_2} \nabla_{E_1} E_2, E_1 \rangle - \langle \nabla_{[E_1, E_2]} E_2, E_1 \rangle$$

$$
\begin{aligned}
&= R(E_1, E_2, E_2, E_1) \\
&= K \\
&= K \, dA(E_1, E_2)
\end{aligned}
$$

✓ $\square$

<u>global Gauss-Bonnet</u>

surface M^2 = <u>triangularizable</u> if $\exists$ open, disjoint $\{T_i\} \subset M$

s.t. ① $T_i \subset$ coord neighborhood

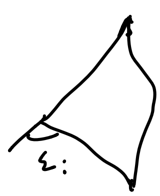
② $M \subset \bigcup \overline{T_i}$

$\overline{T_i}$: closure of T_i

③ each T_i = curved triangle

i.e. ∂T_i = connected, p.w. regular curve with 3 vertices

(ext. angles $\in (-\pi, \pi)$)

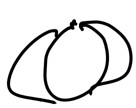


↳ $\{T_i\}$ = triangulation of M

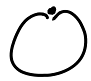
↳ Euler characteristic $= \chi(M)$

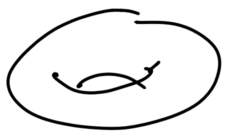
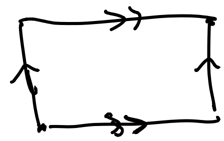
$= (\#\text{ vertices}) - (\#\text{ edges}) + (\#\text{ faces})$

$= V - E + F$

Ex: S^2  $\chi = 1$ pt $- 1$ edge $+ 2$ faces

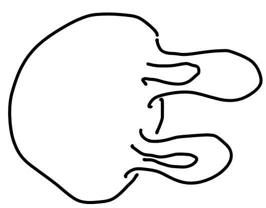
$= 2$

 $\chi = 1\,\text{pt} + 1\,\text{face} = 2$

torus  $=$ 

$\chi = 1\,\text{pt} - 2\,\text{edge} + 1\,\text{face}$
$= 0$

sphere w/ 2 handles ... $\chi = -2$



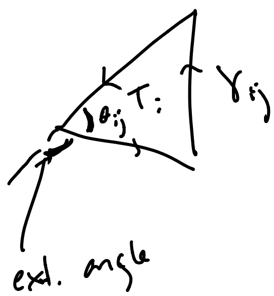
thm: if M^2 is compact, orientable, surface with ∂
$\Rightarrow$ M triangularizable
and χ ind. of triangulation

global Gauss–Bonnet: (M^2, g) = compact, orientable, Riem. surface with boundary

$$\Rightarrow \int_M K + \int_{\partial M} k_N = 2\pi\,\chi(M)$$

proof: let $\{T_i\}$ = triangulation of M
$\{\gamma_{ij} : j = 1, 2, 3\}$ = sides of T_i

$\{\theta_{ij} : j = 1, 2, 3\}$ = interior angles

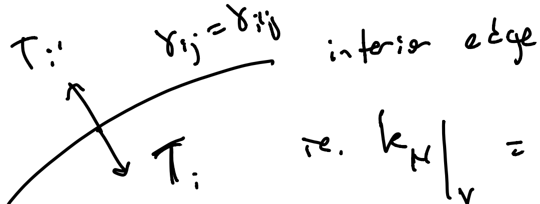


loc. G/B: $\displaystyle\int_{T_i} K \, dA + \int_{\partial T_i} k_N + \underbrace{\sum \text{ext. angles}}_{\sum_{j=1}^{3} \pi - \theta_{ij}} = 2\pi$

$$\sum_i \Bigg(\underbrace{\int_{T_i} K}_{①} + \underbrace{\int_{\partial T_i} k_N}_{②} + 3\pi - \underbrace{\sum_j \theta_{ij}}_{③} \Bigg) = \sum_i 2\pi$$

$$① = \sum_i \int_{T_i} K \, dA = \int_M K \, dA$$

$$② = \sum_i \int_{\partial T_i} k_N \, ds = \int_{\partial M} k_N \, ds$$



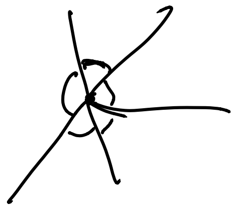
i.e. $k_N\big|_{\gamma_{ij}} = -k_N\big|_{\gamma_{i'j}}$

V = # vertices
E = # edges
F = # faces
= # triangles

$$③ = \sum_i \sum_j \theta_{ij} = 2\pi \, (\text{\# int. vertices}) + \pi \, (\text{\# ext. vertices})$$

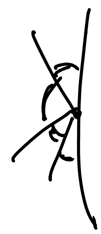
$$= 2\pi V - \pi \, (\text{\# ext. vertices})$$

p = interior vertex



$\sum$ int angles @ p $= 2\pi$

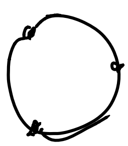
p = exterior vertex



$\sum$ int angles @ p $= \pi$

so far: $$\int_M K\,dA + \int_{\partial M} k_N + \overbrace{3\pi F - 2\pi V + \pi(\#\text{ ext. vertices})}^{-\sum\sum\theta_{ij}}$$
$$= 2\pi F$$

counting: $$3F = 2(\#\text{ interior edges}) + (\#\text{ ext. edges})$$
$$= 2E - (\#\text{ ext. edges})$$
$$= 2E - (\#\text{ ext. vertices})$$



$\boxed{\text{So}}$ $$\int_M K\,dA + \int_{\partial M} k_N = 2\pi(V - E + F)$$
$$= 2\pi\,\chi(M)$$ □

thm (classification of surfaces): if M^2 cpct, oriented, no ∂

$\Rightarrow \chi(M) \in \{2, 0, -2, -4, -6, \dots\}$

and $\chi(M) = \chi(M')$ then $M \cong M'$
homeomorphic

(if M is one-sided $\Rightarrow$ use dble cover)

genus of M $= \dfrac{2-\chi(M)}{2}$ "$=$ # of handles"

Corollary: (M^2, g) cpct, no ∂

① if $M \cong S^2$ or $P^2 = S^2/\{x \sim -x\}$ then $K > 0$ somewhere

② if $M \cong T^2$ or klein bottle (T^2 = dbl cover)

then either $K \equiv 0$ or $K > 0$ and $K < 0$

③ if M any other surface, then $K < 0$ somewhere

proof: M oriented $\Rightarrow$ $\displaystyle\int_M K\, dA = 2\pi\chi(M)$

$M = S^2 \Rightarrow \chi = 2$

$M = T^2 \Rightarrow \chi = 0$

else, $\Rightarrow \chi < 0$

M unorientable

$\Rightarrow$ use dble cover

[aside: if $p: \overline{M} \to M$ covering map, g = metric on M

$\Rightarrow \exists$ metric on $\overline{M}$ st. p = loc. isometry $\square$

Rmk: Kazdan-Warner: given function $\overline{K} : M \to \mathbb{R}$

satisfying required sign of Corollary

$\Rightarrow \exists$ metric g on M st $K_g = \overline{K}$

Cor: $K > 0 \Rightarrow M \cong S^2$ or P^2

$K \leq 0 \Rightarrow M$ has genus ≥ 1

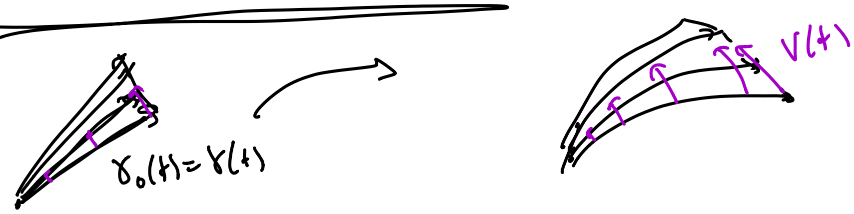
Jacobi fields

consider $F(s,t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$ smooth family of curves

$$F(s,t) = \gamma_s(t)$$

spse each $\gamma_s(t) : [0,1] \to M$ = geodesic

Ex: $\gamma_s(t) = \exp_p(t(v + sw))$



$V(t) = \left.\frac{\partial F}{\partial s}\right|_{s=0}$ = Jacobi field on $\gamma \equiv \gamma_0(t)$

assume F embedding (pretend $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}$ coord fields)

since $t \mapsto \gamma_s(t)$ = geodesic

$$\Rightarrow \frac{D\gamma'}{dt} = \nabla_{\partial_t} \partial_t = 0 \quad \forall t, s$$

$$\Rightarrow \frac{D}{ds}\frac{D}{dt}\gamma' = 0 = \nabla_{\partial_s}\nabla_{\partial_t}\partial_t$$

$$= R(\partial_s, \partial_t)\partial_t + \nabla_{\partial_t}\nabla_{\partial_s}\partial_t$$

$$= R(\partial_s, \partial_t)\,\partial_t + \nabla_{\partial_t}\nabla_{\partial_t}\partial_s$$

@ $s=0 \Rightarrow$ $$\frac{D^2 V}{dt^2} + R(V, \gamma')\gamma' = 0$$

Jacobi equation

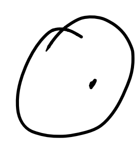
↳ linear, 2^{nd} order ODE for $V(t)$

Ex: $F(s,t) = \exp_p(t(v+sw))$

$$\Rightarrow \left.\frac{\partial F}{\partial s}\right|_{s=0} = \left. D\exp_p\right|_{tv}(tw)$$ = Jacobi field

solves Jacobi eqn

→ to understand $\exp_p$ just need to understand solns to Jacobi eqn



justify computation:

if $f(s_1, \dots, s_k) : \Omega^k \to M$ smooth fn

$(x^1, \dots, x^n) : U^n \to M$ coords near $f(0)$

s.t. $x^i(0) = f(0)$

consider $f_\varepsilon : \Omega \times U \to \Omega \times U$

$(s^\alpha, x^i) \mapsto (\varepsilon s^\alpha, f(s^\alpha) + \varepsilon x^i)$

↳ $$\left. Df_\varepsilon\right|_{(0,0)} = \boxed{\begin{matrix} \varepsilon I & 0 \\ * & \varepsilon I \end{matrix}}$$

⇒ loc. diffeo near $(0,0)$

⇒ induces coords on $\Omega \times M$ near $(0,0)$

and $$\frac{\partial f_\varepsilon}{\partial s^\alpha} \xrightarrow[\varepsilon \to 0]{} \left(0, \frac{\partial f}{\partial s^\alpha}\right)$$

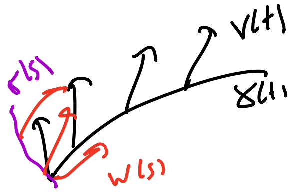
$$\frac{D}{ds^\alpha}\frac{\partial f_\varepsilon}{\partial s^\beta} \xrightarrow[\varepsilon \to 0]{} \left(0, \frac{D}{ds^\alpha}\frac{\partial f}{\partial s^\beta}\right)$$

etc ...

<u>Prop</u>: $\gamma : [0,\ell] \to M$ geodesic

$V : [0,\ell] \to TM$ Jacobi field ⇔ V solves $V'' + R(V,\gamma')\gamma' = 0$

<u>proof</u>: $v, w \in T_{\gamma(0)}M$, ETS: $\exists$ variation $F(s,t)$ thru geodesics s.t. $\frac{\partial F}{\partial s}\Big|_{s=0} = V(t)$ (= Jacobi field) with initial cond $\begin{cases} V(0) = v \\ V'(0) = w \end{cases}$



choose path $\sigma(s)$ s.t. $\sigma(0) = \gamma(0)$, $\sigma'(0) = v$

choose $W(s) \in T_{\sigma(s)}M$ $\quad W(0) = \gamma'(0)$, $\frac{DW}{ds}(0) = w$

set $F(s,t) = \exp_{\sigma(s)}(t W(s))$

$s = 0 \Rightarrow F(0,t) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t)$

$\left.\frac{\partial F}{\partial s}\right|_{s=0}$ = Jacobi field on γ s.t. ⁄ $\square$

take $\gamma : [0,\ell] \to M$ geodesic PBAL

$\{ V : [0,\ell] \to TM$ Jacobi fields $\Leftrightarrow$ V solves $V'' + R(V,\gamma')\gamma' = 0 \}$

E_i = ON basis of $T_{\gamma(0)}M$, $E_n = \gamma'(0)$

$\hookrightarrow$ $E_i(t)$ parallel transport along $\gamma(t)$

$\Rightarrow$ $E_i(t)$ = ON basis of $T_{\gamma(t)}M$, $E_n = \gamma'(t)$

write $\underline{V(t) = \sum_{i=1}^n a^i(t) E_i(t)}$ $\longrightarrow$ $\frac{D^2 V}{dt^2} = \sum_{i=1}^n \frac{d^2 a^i(t)}{dt^2} E_i(t)$

then V = Jacobi field ($\Leftrightarrow$ V solves Jacobi eqn)

$\Leftrightarrow$ $V'' + \underline{R(V,\gamma')\gamma'} = 0$

$\Leftrightarrow$ $\frac{d^2 a^i}{dt^2} + \sum_{j=1}^n a^j \underbrace{R(E_j,\gamma',\gamma',E_i)}_{A_{ij}} = 0$

$\Leftrightarrow$ $\frac{d^2 a^i}{dt^2} + \sum_{j=1}^n A_{ij}(t)\, a^j = 0$ (**)

note: $E_n = \gamma'$ $\Rightarrow$ $A_{in} = R(E_n,\gamma',\gamma',E_i) = R(\gamma',\gamma',\gamma',E_i) = 0$

$A_{ni} = A_{nn} = 0$

i.e. (**) $\Leftrightarrow$ $\begin{cases} \frac{d^2 a^i}{dt^2} + \sum_{j=1}^{n-1} A_{ij} a^j = 0 & i = 1,\dots,n-1 \\ \frac{d^2 a^n}{} = 0 & \end{cases}$

$V = \sum a^i E_i$

$) \, dt^2$

i.e. if $f(t) = \langle V, \gamma' \rangle \Rightarrow f'' = \langle V'', \gamma' \rangle$
$$= -R(V, \gamma', \gamma', \gamma') = 0$$

$$\Rightarrow f(t) = a + bt$$

prop: for any $v, w \in T_{\gamma(0)} M$
$\Rightarrow \exists!$ Jacobi field $\underset{V_{v,w}}{\overset{\shortparallel}{V}} : [0, \ell] \to TM$ st. $\begin{cases} V(0) = v \\ V'(0) = w \end{cases}$

and $(v, w) \mapsto V_{v,w}(t)$ linear isomorphism

(so, space of Jacobi fields on γ = $2n$-dim)

prop: if $V(t)$ Jacobi field, $\begin{matrix} V(0) \perp \gamma'(0) \\ V'(0) \perp \gamma'(0) \end{matrix} \Rightarrow V(t) \perp \gamma'(t) \ \forall t$

or $\begin{matrix} V(t_1) \perp \gamma'(0) \\ V(t_2) \perp \gamma'(0) \end{matrix} \Rightarrow V(t) \perp \gamma'(t) \ \forall t$

(and $(\alpha + \beta t)\gamma'$ = Jacobi field)

Jacobi fields in const curvature

M has $\sec \equiv k$ = const
$\Rightarrow R(X, Y, Z, W) = k \left(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \right)$

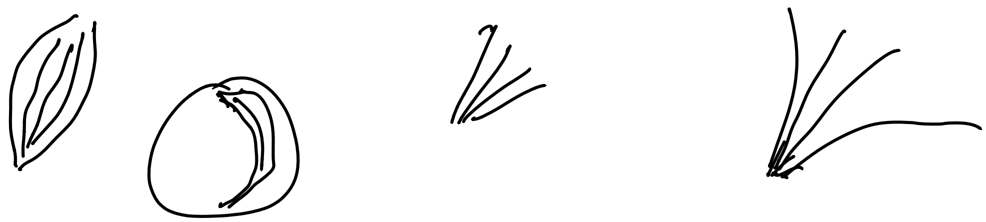
$\Rightarrow A_{ij} = R(E_j, \gamma', \gamma', E_i)$
$= k \delta_{ij}$ if $i, j \neq n$ $(E_n = \gamma')$

so $V = \sum_{i=1}^{n} a^i E_i$ Jacobi field

$$\iff \frac{d^2 a^i}{dt^2} + k a^i = 0 \quad (i = 1, \dots, n-1), \quad \frac{d^2 a^n}{dt^2} = 0$$

$$\begin{cases} k > 0 \\ k = 0 \\ k < 0 \end{cases} \text{ gives solns } \begin{cases} \sin(\sqrt{k}\, t), \ \cos(\sqrt{k}\, t) \\ 1, \ t \\ \sinh(\sqrt{-k}\, t), \ \cosh(\sqrt{-k}\, t) \end{cases}$$

positive curvature   zero k   negative curvature



generalized sine $= \sin_k(t)$

$$= \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, t) & k > 0 \\ t & k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, t) & k < 0 \end{cases}$$

generalized polar coords

$(r, \theta) \mapsto \exp_p(r\theta)$ (identify $T_pM \cong \mathbb{R}^n$ via isometry)

$\in (0, R) \times S^{n-1}$

we saw: metric $g = dr^2 + r^2 f_{ij}\, d\theta^i d\theta^j$

$\hookrightarrow$ where $g(\partial_{\theta^i}, \partial_{\theta^j}) = \langle D\exp_p|_{r\theta}(r\partial_{\theta^i}),$

$D\exp_p|_{r\theta}(r\partial_{\theta^j})\rangle$

etc.

$\mathrm{Sect} = k \Rightarrow D\exp_p|_{r\theta}(rw) = $ Jacobi field $V(r)$

along $\gamma(r) = \exp_p(r\theta)$

initial cond $\begin{cases} V(0) = 0 \\ V'(0) = w \end{cases}$

if $w \perp \theta \Rightarrow V(r) = A\sin_k(r) + \cancel{B\cos_k(r)}$

$= w\sin_k(r)$

$\boxed{\text{So}}$ $g(\partial_{\theta^i}, \partial_{\theta^j}) = \sin_k^2(r)\, g_{S^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j})$

$\Rightarrow g = dr^2 + \sin_k(r)^2 g_{S^{n-1}}$

$[\ n=2 \Rightarrow g = dr^2 + \sin_k(r)^2 d\theta^2$

OR if $(x^1, \dots, x^n) \mapsto \exp_p(\sum x^i E_i)$ = normal coords

$\mathrm{Sect} \equiv k$, $x \neq 0$

$\longrightarrow$ $g|_x(v,v) = \dfrac{(v\cdot x)^2}{|x|^2} + \dfrac{\sin_k^2|x|}{|x|^2}\left|v - \dfrac{(v\cdot x)x}{|x|^2}\right|^2$

euclidean dot product

Cor: if $M^n, \widetilde{M}^n$ with $\mathrm{Sec} \equiv k \Rightarrow M, \widetilde{M}$ loc. isometric

proof: $p \in M$, $\widetilde{p} \in \widetilde{M}$, choose normal coords $x^i, \widetilde{x}^i$ in $B_r(p), B_r(\widetilde{p})$

$\Rightarrow$ pullback metrics $g, \widetilde{g}$

$g = \widetilde{g}$ on $B_r(0) \setminus \{0\}$

$\Rightarrow g = \widetilde{g}$ on $B_r(0)$

pick any isometry $I : T_pM \to T_{\widetilde{p}}\widetilde{M}$

$\Rightarrow$ isometry $= \exp_{\widetilde{p}} \circ I \circ \exp_p^{-1} : B_r(p) \to B_r(\widetilde{p})$

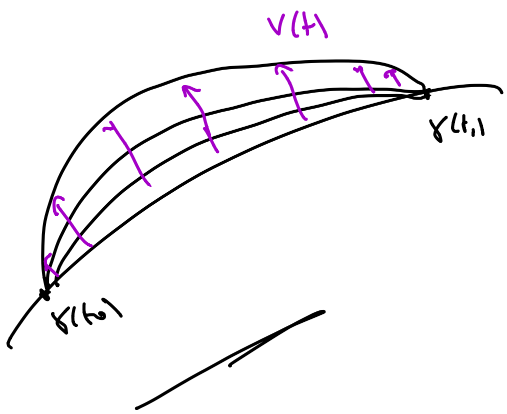
$\square$

Conjugate points

$\gamma : [0, l] \to M$ geodesic in (M, g)

$\gamma(t_0), \gamma(t_1)$ conjugate along γ $\Leftrightarrow$ $\exists$ non-zero Jacobi field $V : [t_0, t_1] \to TM$ on γ s.t. $V(t_0) = V(t_1) = 0$

$t_0 \neq t_1$



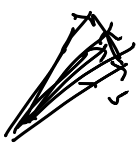
__Prop^n:__ $\gamma(t) = \exp_p(tv)$

then $D\exp_p\big|_{t_1 v}$ = non-singular

$\iff \gamma(0), \gamma(t_1)$ __not__ conjugate

__proof:__ recall: given any $w \in T_pM$

$$V(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(t(v+sw)) = \text{Jacobi field}$$



$$= \underline{\underline{D\exp_p\big|_{tv}(tw)}} \qquad \text{solves } \begin{cases} V(0) = 0 \\ V'(0) = w \end{cases}$$

$\gamma(t_1)$ conjugate to $\gamma(0)$

$\iff \exists$ Jacobi field $V : [0, t_1] \to TM$ on γ

st $\underline{V(0) = 0} = V(t_1)$, $V \not\equiv 0$

$\hookrightarrow \underline{V'(0) =: w} \neq 0$

$\iff V(t) = D\exp_p\big|_{tv}(tw)$ by uniqueness of ODEs

and $V(t_1) = 0$

$\iff D\exp_p\big|_{t_1 v}(t_1 w) = 0$ (and $t_1 \neq 0$)

$\iff D\exp_p\big|_{t_1 v}$ singular $\square$

__Remark:__ $\gamma(t_0), \gamma(t_1)$ __not__ conjugate

$\Leftrightarrow$ can solve (uniquely) the boundary-value problem: $V : [t_0, t_1] \to TM$ Jacobi field on γ

s.t. $V(t_0) = v_0$
$V(t_1) = v_1$ for prescribed v_0, v_1

idea of proof: $v \in T_{\gamma(t_0)}M$, solve V_v = Jacobi field

s.t. $\begin{cases} V_v(0) = 0 \\ V_v'(0) = v \end{cases}$

$v \mapsto V_v(t_1)$

(not conj) = linear isomorphism

using Jacobi fields to say stuff about curvature:

$\hookrightarrow$ idea: if sect $\leq k$ (resp. $\geq k$)

$\Rightarrow$ conjugate pts spaced at least as far as in model space with sect $\equiv k$

(resp. at most)

Non-positive curvature

Cartan-Hadamard thm: (M, g) complete, simply-connected. sectional curvature ≤ 0

$\Rightarrow \forall p \in M$, $\exp_p : T_pM \to M$ = diffeo

and $d(\exp_p(v), \exp_p(w)) \geq |v - w|$

Cor: if M^n complete, $K \le 0$

then $M = \tilde{M}/\Gamma$ $\quad \tilde{M}$ diffeo to $\mathbb{R}^n$,

$\Gamma = \pi_1(M)$

proof of cor: take $\tilde{M}$ = universal cover

$p : \tilde{M} \to M$ covering map $\Rightarrow$ let $\tilde{g} = p^* g$

$\Rightarrow (\tilde{M}, \tilde{g})$ complete (geodesics in M lift to geodesics in $\tilde{M}$)

and $\tilde{K} \le 0$

$\Rightarrow \tilde{M}$ diffeo to $\mathbb{R}^n$ ((Cartan-Hadamard) $\square$

we will in fact show $\exp_p$ = covering map (even if M not simply-connected)

proof of Cartan-Hadamard:

$\exp_p : T_pM \to M$ defined on all of T_pM (Hopf-Rinow)

$\gamma(t) = \exp_p(tv)$ = geodesic, $|v| = 1$

recall: $V(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p(t(v+sw)) = D\exp_p\big|_{tv}(tw)$

Jacobi field on γ

s.t. $V(0) = 0$, $V'(0) = w$ $\left(\neq 0^{\text{wlog}}\right)$

define $f(t) = |v(t)| = \langle v(t), v(t)\rangle^{1/2}$

↳ f continuous, and smooth on $\{v \neq 0\}$

claim: $f'' \geq 0$ on $\{v \neq 0\}$

$v'' + R(v,\gamma')\gamma' = 0$

$$f' = \frac{\langle v', v\rangle}{|v|}$$

$$f'' = \frac{\langle v', v'\rangle}{|v|} - \frac{\langle v', v\rangle^2}{|v|^3} + \frac{\langle v'', v\rangle}{|v|}$$

$$= \frac{1}{|v|}\left(\underbrace{|v'|^2 - \frac{\langle v', v\rangle^2}{|v|^2}}\right) - \frac{\langle R(v,\gamma')\gamma', v\rangle}{|v|}$$

$$= \frac{1}{|v|}\left|v' - \frac{\langle v, v'\rangle v}{|v|^2}\right|^2 - \frac{1}{|v|}\underline{R(v,\gamma',\gamma',v)}$$

$$= \underbrace{\frac{1}{|v|}|\cdots|^2}_{\geq 0} \underbrace{- \frac{1}{|v|}|v\wedge\gamma'|^2 \underbrace{K(v,\gamma')}_{\text{sect} \leq 0}}_{\geq 0}$$

≥ 0

and $v(0) = 0$, $v'(0) = w \Rightarrow v(t) = tw + O(t^2)$

↳ $v \neq 0$ for $\underline{t \in (0, \varepsilon)}$

and $$f'(t) = \frac{\langle v', v\rangle}{|v|} = \frac{\langle tw + O(t^2), w + O(t)\rangle}{|w|t + O(t^2)}$$

$$= |w| + O(t)$$

$\longrightarrow |w|$ as $t \to 0$

and $f'' \geq 0$

$\Rightarrow f'(t) \geq |w| \quad \forall\, t \in (0, \varepsilon)$

$\Rightarrow |V| \neq 0 \quad \forall\, t > 0$ and $f'(t) \geq |w|$

$\Rightarrow f(t) = |V(t)| \geq |w|\, t$

(ie. γ has no conjugate pts)

$\Rightarrow |D\exp_p|_{tv}(w)| \geq |w|$

so $\exp_p$ = loc. diffeo, loc. dist increasing

define $\bar{g} = \exp_p^* g \quad \Rightarrow \exp_p : (T_pM, \bar{g}) \longrightarrow (M, g)$

= loc. isometry

and complete since rays thru origin exist $\forall\, t$, PBAL

lemma: $f : (M_1, g_1) \longrightarrow (M_2, g_2)$ loc. isometry $(f^* g_2 = g_1)$

and M_1 complete, M_2 connected

$\Rightarrow f$ covering map

lemma $\Rightarrow \exp_p : T_pM \to M$ covering map

M simply-connected $\Rightarrow M$ = universal cover of M

$\Rightarrow \exp_p$ = diffeo

|

$\hookrightarrow \quad L(\exp_p \circ \gamma) = \int_0^\ell |D\exp_p \gamma'| \geq \int_0^\ell |\gamma'| = L\gamma$

$\Rightarrow \exp_p$ dist increasing

$\square$

<u>proof of lemma</u>: observations

1. $\underline{\underline{|Df|_p v| = |v|}} \quad \Rightarrow Df|_p$ non-singular

   $\Rightarrow f$ loc. diffeo

2. $d(f(p), f(\tilde{p})) \leq d(p, \tilde{p})$

   γ = minim geodesic $p \longrightarrow \tilde{p}$ in M_1

   then $d(p, \tilde{p}) = L\gamma = L f\circ\gamma \geq d(f(p), f(\tilde{p}))$

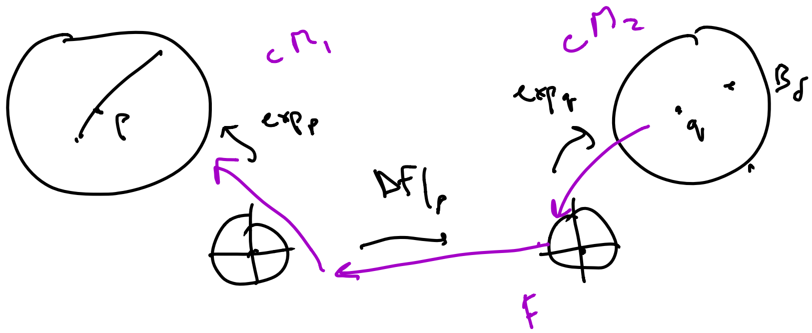
3. if γ geodesic in $M_1 \Rightarrow f\circ\gamma$ geodesic in M_2 (HW)

take $q = f(p)$, choose $\delta > 0$ st $\exp_q : B_\delta(0) \rightarrow B_\delta(q) \subset M_2$

= diffeo

$\hookrightarrow$ define $F = \underline{\exp_p} \circ \underline{Df|_p^{-1}} \circ \underline{\exp_q^{-1}} : B_\delta(q) \rightarrow M_1$

F well-defined



<u>claim</u>: $f(F(x)) = x$ for $x \in B_\delta(q) \subset M_2$

first, $f(F(q)) = q$

take $v \in T_pM_1$, $|v| < \delta$ $\Rightarrow$ $\exp_p(tv) = \gamma(t)$

= geodesic

$\Rightarrow$ $\tilde{\gamma}(t) = f \circ \gamma(t)$ = geodesic in M_2

s.t. $\tilde{\gamma}(0) = q$

$\tilde{\gamma}'(0) = Df|_p\, v$

$\Rightarrow$ $\tilde{\gamma}(t) = \exp_q(t\, Df|_p\, v)$

$\Rightarrow$ $f(\exp_p(v)) = \underline{\exp_q}(\underline{D}f|_p\, v)$

and $\underbrace{|Df|_p^{-1} \exp_q^{-1}(x)|}_{=v} = |\exp_q^{-1}(x)| < \delta$

$x \in B_\delta(q)$

$\Rightarrow$ $f(F(x)) = x$ $\square$

$f(M_1)$ open, nonempty in M_2

<u>claim</u>: $f(M_1)$ closed ($\Rightarrow f(M_1) = M_2$)

let $\hat{q} \in \overline{f(M_1)}$

choose $\delta > 0$ s.t. $\exists\, \underset{\substack{\| \\ f(p)}}{q} \in B_{\underline{\delta/2}}(\hat{q}) \cap f(M_1)$

<u>and</u> $\exp_q|_{B_\delta(0)}$ = diffeo

$F = \exp_p \circ Df|_p^{-1} \circ \exp_q^{-1} : \underline{B_\delta(q)} \to M_1$

claim 1 $\Rightarrow$ $f(F(x)) = x$

and $\tilde{q} \in B_\delta(q) \Rightarrow f(F(\tilde{q})) = \tilde{q}$

$\Rightarrow \tilde{q} \in f(M_1)$ $\square$

So: f = loc diffeo, surjective

↳ NTS: uniform covering property

take $q \in M_2$, choose $\delta > 0$ s.t. $\exp_{\tilde{q}}|_{B_{2\delta}(0)}$ = diffeo

$\forall \tilde{q} \in B_\delta(q)$

claim: $f^{-1} B_\delta(q) = \bigcup_{p \in f^{-1}(q)} B_\delta(p)$ disjoint union

and $f|_{B_\delta(p)}$ = diffeo by claim 1

spse $f(\tilde{p}) = \tilde{q} \in B_\delta(q)$

$\Rightarrow \underbrace{f \circ \exp_{\tilde{p}} \circ Df|_{\tilde{p}}^{-1} \circ \exp_{\tilde{q}}^{-1}}_{= F}$ = id on $B_{2\delta}(\tilde{q})$

(claim 1)

$\Rightarrow f(F(q)) = q$ and $d(F(q), \tilde{p}) = d(q, \tilde{q}) < \delta$

$\Rightarrow \tilde{p} \in B_\delta(p)$ for some $p \in f^{-1}(q)$

$\Rightarrow f^{-1} B_\delta(q) \subset \bigcup_{p \in f^{-1}(q)} B_\delta(p)$

since f dist decreasing $\Rightarrow f(B_\delta(p)) \subset B_\delta(f(p) = q)$

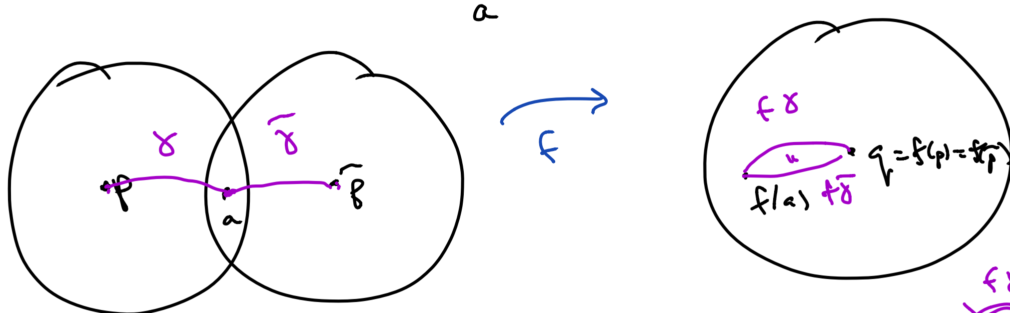
$p \in f^{-1}(q)$

$$\Rightarrow \bigcup_{p \in f^{-1}(q)} B_\delta(p) \subset f^{-1} B_\delta(q)$$

$\Rightarrow$ (=)

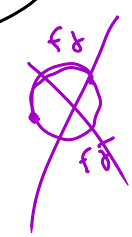
NTS: balls $B_\delta(p)$, $p \in f^{-1}(q)$ disjoint

Spse not disjoint: $B_\delta(p) \cap B_\delta(\tilde{p}) \ni a$, $p, \tilde{p} \in f^{-1}(q)$



choose geodesic PBAL $\gamma : p \to a$, inside $B_\delta(p)$

$\tilde{\gamma} : \tilde{p} \to a$, inside $B_\delta(\tilde{p})$



ie $\gamma(t) = \exp_p(tv)$ for some v...

$f \circ \gamma$, $f \circ \tilde{\gamma}$ geodesic in $B_\delta(q)$, PBAL

s.t. $= q$ @ 0 and pass thru $f(a)$

since $\exp_q|_{B_\delta} : B_\delta(0) \to B_\delta(q)$ = diffeo

$\Rightarrow f \circ \gamma(t) = f \circ \tilde{\gamma}(t)$

since $f|_{B_\delta(p)}$, $f|_{B_\delta(\tilde{p})}$ diffeo $\Rightarrow \exists\, t_a$ st

$\gamma(t_a) = \tilde{\gamma}(t_a) = a$

and f = loc. isometry

$\hookrightarrow \gamma = f^{-1}\sigma(t)$

$\tilde{\gamma} = f^{-1}\sigma(t)$

and $\gamma'(t_0) = \tilde{\gamma}'(t_0)$

$\Rightarrow \gamma = \tilde{\gamma}$ by uniqueness of geodesic

$\Rightarrow p = \gamma(0) = \tilde{\gamma}(0) = \tilde{p}$ $\square$

<u>Const curvature revisited</u>

(M,g) complete, $\underline{K \equiv k = \text{const}}$

$p \in M$, $v, w \in T_pM$, $\underline{\underline{|w| = 1}}$ $\rightsquigarrow$ $\exp_p(t(v+sw))$

$\hookrightarrow V(t) = D\exp_p\big|_{tv}(tw)$ = Jacobi field along $\gamma(t) = \exp_p(tv)$

$V(0) = 0$, $\underline{V'(0) = w}$

$\hookrightarrow V(t) = \sum_i a^i E_i(t)$ $\quad E_i$ parallel, ON, $E_n = \gamma'$

$$\Rightarrow \begin{cases} (a^i)'' + k a^i = 0 & \underline{i = 1, \dots, n-1} \\ (a^n)'' = 0 \end{cases}$$

if $w \perp v$ $\Rightarrow V(t) = w(t)\,\text{sn}_k(t)$

$w(t)$: parallel transport of w $\quad$ sn_k: gen. sine $\begin{cases} \frac{\sin(\sqrt{k}\,t)}{\sqrt{k}} \\ t \\ \frac{\sinh(\sqrt{-k}\,t)}{\sqrt{-k}} \end{cases}$

$$\Rightarrow \underline{|D\exp_p\big|_{tv}(tw)| = |w|\,|\text{sn}_k(t)|}$$

$|v| = 1$

if $w \parallel v$ $\Rightarrow V(t) = w(t)\,t$

$$\Rightarrow \underline{\left|D\exp_p\big|_{tv}(tw)\right| = |w||t|}$$

<u>lemma:</u> if $\tilde{M}$ also complete, $K \equiv k$, $\tilde{p} \in \tilde{M}$, $\exp_p\big|_{B_r(0)}$ diffeo

$I : T_pM \to T_{\tilde{p}}\tilde{M}$ = linear isometry

then $f = \exp_{\tilde{p}} \circ I \circ \exp_p^{-1} : B_r(p) \longrightarrow B_r(\tilde{p})$

= isometry

<u>proof:</u> $w \in T_pM$, $w \perp v$

$$\Rightarrow \left|D\exp_p\big|_v(w)\right| = \left|D\exp_p\big|_{\frac{v}{|v|}|v|}(w|v|)\right|\frac{1}{|v|}$$

$$= \frac{\sin_k|v|}{|v|}|w|$$

and $I(w) \perp I(v)$

$$\Rightarrow \left|D\exp_{\tilde{p}}\big|_{I(v)} I(w)\right| = \frac{\sin_k|I(v)|}{|I(v)|}|I(w)|$$

$$= \frac{\sin_k|v|}{|v|}|w|$$

$$\Rightarrow \left|Df\big|_{\exp_p(v)} W\right| = |W|$$

if $w \parallel v \Rightarrow I(w) \parallel I(v)$

$$\Rightarrow \left|D\exp_p\big|_v(w)\right| = |w|$$
$$= |I(w)| = \left|D\exp_{\tilde{p}}\big|_{I(v)} I(w)\right| \qquad \square$$

Classification of spaceforms: (M^n, g) simply-connected, complete, $K \equiv k$ $(n \geq 2)$

then $k = \begin{cases} -1 \\ 0 \\ 1 \end{cases} \Rightarrow (M, g)$ isometric to $\begin{cases} \mathbb{H}^n \\ \mathbb{R}^n \\ \mathbb{S}^n \end{cases}$

proof: note $\mathbb{H}^n, \mathbb{R}^n, \mathbb{S}^n$ complete, simply-connected, $K = -1, 0, 1$

$k=-1$: $p \in \mathbb{H}^n \Rightarrow \exp_p : \underline{T_p \mathbb{H}^n} \longrightarrow \mathbb{H}^n$ diffeo^s

$\tilde{p} \in M$ Cartan Hadamard $\exp_{\tilde{p}} : \underline{T_{\tilde{p}} M} \longrightarrow M$

pick $I : T_p \mathbb{H}^n \longrightarrow T_{\tilde{p}} M$ linear isometry

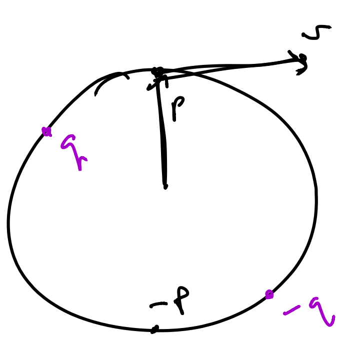
$\Rightarrow f = \exp_{\tilde{p}} \circ I \circ \exp_p^{-1} : \mathbb{H}^n \longrightarrow M$ isometry

$k=0$: same proof

$k=1$: build $f : S^n \longrightarrow M$ loc. isometry

$p \in S^n$ $\qquad \exp_p(tv) = p \cos t + v \sin t$

$v \in T_p S^n$, $|v| = 1$ $\qquad \Rightarrow \exp_p : B_\pi(0) \longrightarrow S^n \setminus \{-p\}$



check: $p \cos t + v \sin t = p \cos t' + v' \sin t'$

$\Rightarrow p \underbrace{(\cos t - \cos t')}_{=0} = \underbrace{v' \sin t' - v \sin t}_{\perp p}$

$\Rightarrow t = t'$ since $t', t \in [0, \pi)$

$\Rightarrow v = v'$

aside: $|p\cos t + v\sin t|^2 = |p|^2\cos^2 t + |v|^2 \sin^2 t = 1$

$\frac{d^2}{dt^2} \exp_p(tv) = -(p\cos t + v\sin t) \perp T_{\exp_p(tv)} S^n$

$I : T_p S^n \to T_{\tilde{p}} M$ linear isometry

$\hookrightarrow f_p : S^n \setminus \{-p\} \to M$
$= \exp_{\tilde{p}} \circ I \circ \exp_p^{-1}$ $\Big\}$ = loc. isometry

choose $q \in S^n \setminus \{p, -p\}$

$\hookrightarrow$ define $f_q : S^n \setminus \{-q\} \to M$
$= \exp_{f_p(q)} \circ Df_p\big|_q \circ \exp_q^{-1}$ $\Big\}$ = loc isometry

$Df_p\big|_q : T_q S^n \to T_{f(q)} M$ isometry

and $f_q(q) = f_p(q)$

and $Df_q\big|_q = Df_p\big|_q$

and $S^n \setminus \{-p, -q\}$ connected

$\Rightarrow f_p = f_q$ on $S^n \setminus \{-p, -q\}$

define $f : S^n \to M$ by

$$f(x) = \begin{cases} f_p(x) & x \neq -p \\ f_q(x) & x \neq -q \end{cases}$$

smooth, loc. isometry

$\Rightarrow f =$ covering map

$+ M$ simply-connected $\Rightarrow f$ diffeo $\square$

$\Uparrow$

($f =$ homeo + loc. diffeo)

general curvature

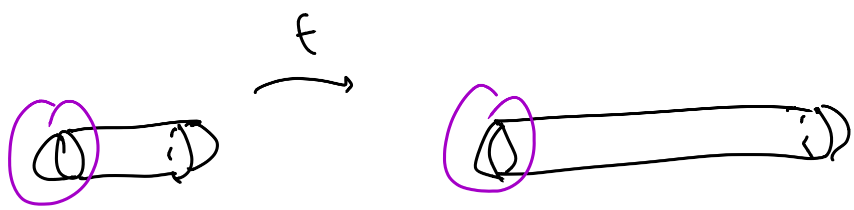
Q: if curvatures of $M, \tilde{M}$ are "the same" $\overset{?}{\Rightarrow}$ $M, \tilde{M}$ isometric

eq Q: if $f : (M, g) \to (\tilde{M}, \tilde{g}) =$ diffeo

s.t. $\tilde{R}(Df X, Df Y, Df Z, Df W) = R(X, Y, Z, W)$

$\overset{?}{\Rightarrow} f =$ isometry?

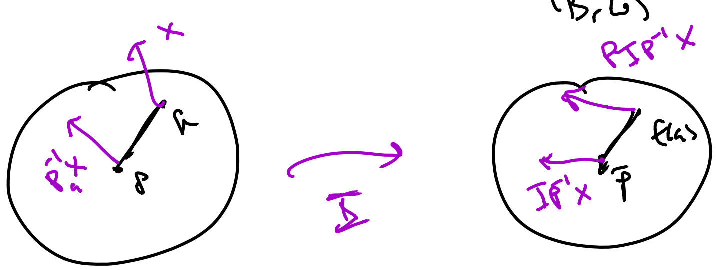
NO



isometry near caps but stretches cylinder

"correct version": spse $p \in M$, $\tilde{p} \in \tilde{M}$

$\exp_p|_{B_r(0)}$, $\exp_{\tilde{p}}|_{B_r(0)}$ diffeo



I = linear isometry $T_pM \to T_{\tilde p}\tilde M$

define $f = \exp_{\tilde p} \circ I \circ \exp_p^{-1} : B_r(p) \to B_r(\tilde p)$

define $\varphi_q : T_qM \longrightarrow T_{f(q)}\tilde M$

$$\varphi_q = \underset{\substack{\uparrow \\ \text{parallel transport along geodesic ray } \tilde p \to f(q)}}{P_{f(q)}} \circ I \circ P_q^{-1}$$

<u>Thm (Ambrose-Hicks)</u>: if $\tilde R(\varphi_q X, \varphi_q Y, \varphi_q Z, \varphi_q W) = R(X,Y,Z,W)$ $\forall q \in B_r(p)$ & $X,Y,Z,W \in T_qM$

$\Rightarrow f$ = isometry

<u>proof:</u> choose E_i ON basis of T_pM

$\hookrightarrow \tilde E_i = I(E_i)$ = ON basis of $T_{\tilde p}\tilde M$

let $\gamma(t) = \exp_p(tv)$, $|v| = 1$

$\hookrightarrow \tilde\gamma(t) := (f\circ\gamma)(t) = \exp_{\tilde p}(t\,I(v))$ by uniqueness of geodesics

$E_i(t)$, $\tilde E_i(t)$ parallel transport along γ, $\tilde\gamma$ (resp)

$V(t) = D\exp_p\big|_{tv}(tw)$ = Jacobi field along γ s.t. $V(0) = 0$, $V'(0) = w$

$\hookrightarrow V = \sum a^i E_i \Rightarrow \dfrac{d^2 a^i}{dt^2} + a^j A_{ij} = 0$, $a^i(0) = 0$

$\dfrac{da^i}{dt}(0) = w^i = \langle w, E_i\rangle$

$A_{ij} = R(E_j, \gamma', \gamma', E_i)$

similarly for $\tilde{V}(t) = D\exp_{\tilde{p}}\big|_{tI(w)}(tI(w))$

$\varphi_t \equiv \varphi_{\gamma(t)}$

↳ if $\tilde{V} = \sum \tilde{a}^i \tilde{E}_i$

then $0 = \dfrac{d^2\tilde{a}^i}{dt^2} + \tilde{a}^j \tilde{A}_{ij}$ for $\tilde{A}_{ij} = \tilde{R}(\tilde{E}_j, \tilde{\gamma}', \tilde{\gamma}', \tilde{E}_i)$

$= \tilde{R}(\varphi_t E_j, \varphi_t \gamma', \varphi_t \gamma', \varphi_t E_i)$

$\Rightarrow \quad 0 = \dfrac{d^2\tilde{a}^i}{dt^2} + \tilde{a}^j A_{ij}$

$= R(E_j, \gamma', \gamma', E_i)$

$= A_{ij}$

and $\tilde{a}^j(0) = 0$

$\dfrac{d\tilde{a}^j}{dt}(0) = \langle I(w), \tilde{E}_j \rangle = w^j$

so $a^j, \tilde{a}^j$ solve same IVP $\Rightarrow a^j = \tilde{a}^j$

$\Rightarrow |D\exp_p|_{tv}(tw)|^2 = |V(t)|^2 = \sum a^{i\,2}$

$= \sum \tilde{a}^{i\,2} = |D\exp_{\tilde{p}}|_{tI(v)}(tI(w))|^2$

$\Rightarrow |Df|_q(w)| = |w|$ for $q = \exp_p(tv)$ ∎

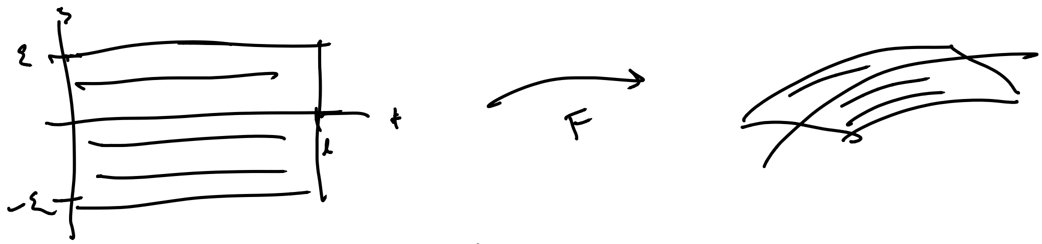
Second variation of length

take $\gamma(t): [0,\ell] \to M$ smooth curve PBAL

↳ $\gamma_s(t) = F(s,t): (-\varepsilon,\varepsilon)\times[0,\ell] \to M$

$=$ 1-parameter family of smooth curves s.t. $\gamma_0 = \gamma$

$\longrightarrow V = \dfrac{\partial F}{\partial s}\Big|_{s=0} =$ variation field on γ



recall length $L\gamma_s = \int_0^\ell |\gamma_s'|\,dt = \int_0^\ell \left|\frac{\partial F}{\partial t}\right| dt$

<u>first variation</u> $= \delta_V L\gamma$

$$= \frac{d}{ds}\Big|_{s=0} L\gamma_s = \int_0^\ell \frac{\langle \nabla_{\frac{\partial F}{\partial s}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \rangle}{\left|\frac{\partial F}{\partial t}\right|}\, dt$$

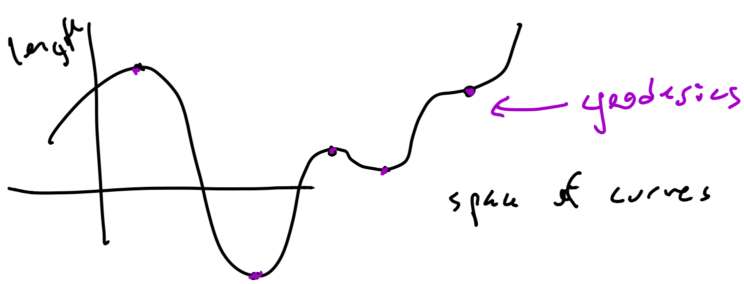
@ $s=0$ $$= \int_0^\ell \langle \nabla_{\partial_t} \partial_s, \partial_t \rangle\, dt$$

$$= \langle V, \gamma' \rangle \Big|_{t=0}^{\ell} - \int_0^\ell \langle V, \frac{D\gamma'}{dt} \rangle\, dt$$

variation = <u>proper</u> if $V(0) = V(\ell) = 0$

↳ $\delta_V L\gamma = 0$ $\forall$ proper variations $\Leftrightarrow$ γ = geodesic

but geodesics need not minimize distance



→ use second variation to detect nature of critical pt second variation $= \delta^2_{V,V} L\gamma$

$$= \frac{d^2}{ds^2}\Big|_{s=0} L\gamma_s$$

$$= \frac{d}{ds}\Big|_{s=0} \int_0^\ell \frac{\langle \nabla_s \partial_t, \partial_t \rangle}{|\partial_t|}\, dt$$

$$= \int_0^\ell \frac{\langle \nabla_s \nabla_s \partial_t, \partial_t \rangle}{|\partial_t|} + \frac{|\nabla_s \partial_t|^2}{|\partial_t|} - \frac{\langle \nabla_s \partial_t, \partial_t \rangle^2}{|\partial_t|^3}\, dt$$

@ $s=0$ using torsion free

$$= \int_0^\ell \langle \nabla_s \nabla_t \partial_s, \partial_t \rangle + |\nabla_t \partial_s|^2 - \langle \nabla_t \partial_s, \partial_t \rangle^2$$

$$= \int_0^\ell R(\partial_s, \partial_t, \partial_s, \partial_t) + \langle \nabla_t \nabla_s \partial_s, \partial_t \rangle + |(\nabla_t \partial_s)^\perp|^2 \; dt$$

where $w^\perp = w - \langle w, \gamma' \rangle \gamma'$

assume γ geodesic $\Longrightarrow$ $\frac{D}{dt}(w^\perp) = \frac{Dw}{dt} - \langle \frac{Dw}{dt}, \gamma' \rangle \gamma'$

$\hookrightarrow \frac{D\gamma'}{dt} = 0$

$$= \left(\frac{Dw}{dt}\right)^\perp$$

and $R(V, \gamma', \gamma', V) = R(V^\perp, \gamma', \gamma', V^\perp)$

by anti-symmetry

so

$$= \int_0^\ell \left|\frac{D}{dt} V^\perp\right|^2 - R(V, \gamma', \gamma', V)\, dt + \langle \nabla_V V, \gamma' \rangle \Big|_{t=0}^{\ell}$$

$$= I_\ell(V^\perp, V^\perp) + \langle \nabla_V V, \gamma' \rangle \Big|_{t=0}^{\ell}$$

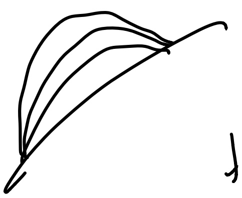
for $\displaystyle I_\ell(V,W) = \int_0^\ell \left\langle \frac{DV}{dt}, \frac{DW}{dt} \right\rangle - R(V,\gamma',\gamma',W)\, dt$

= <u>index form</u> (= symmetric, bilinear)

<u>Note:</u> (variation = proper if $\gamma_s(0) = \gamma_0(0)$, $\gamma_s(\ell) = \gamma_0(\ell)$ $\forall s$)

$\Downarrow$

$V(0) = V(\ell) = 0$ and $\nabla_V V\big|_0 = \nabla_V V\big|_\ell = 0$



if variation proper

then $\displaystyle \frac{d^2}{ds^2}\Big|_{s=0} L\gamma_s = I_\ell(V^\perp, V^\perp)$

$$= \int_0^\ell \left|\frac{D}{dt} V^\perp\right|^2 - R(V^\perp, \gamma', \gamma', V^\perp)\, dt$$

$$= -\int \langle V^\perp, \mathcal{L} V^\perp \rangle$$

where $\mathcal{L}V = V'' + R(V,\gamma')\gamma'$

= Jacobi operator

<u>Note:</u> $\gamma : [0,\ell] \to M$ PBAL, $V : [0,\ell] \to TM$, $V(t) \in T_{\gamma(t)}M$

$\Rightarrow \exists$ variation $F(s,t) = \gamma_s(t) : (-\varepsilon, \varepsilon) \times [0,\ell] \to M$

s.t. $\gamma_0 = \gamma$, $\partial_s F = V$, $\nabla_{\partial_s} \partial_s = 0$

$\hookrightarrow$ set $F(s,t) = \exp_{\gamma(t)}(sV(t))$

(move along $s \longrightarrow$ move along geodesic)

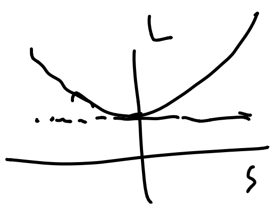
(if $V(0) = 0 = V(\ell)$, then $F(s,t)$ is proper)

Note: if $\gamma : [0,\ell] \to M$ minz, PBAL

i.e. $d(\gamma(0), \gamma(\ell)) = L\gamma = \ell$

if $\gamma_s(t)$ = proper variation (fixes endpts)

then $L\gamma_s(t) \geq d(\gamma_s(0), \gamma_s(\ell)) = L\gamma_0$



$$\Rightarrow \left.\frac{d}{ds}\right|_0 L\gamma_s = 0$$

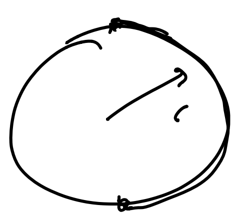
$$0 \leq \left.\frac{d^2}{ds^2}\right|_{s=0} L\gamma_s = I_\ell(V^\perp, V^\perp)$$

Positive curvature

Bonnet–Myers: (M^n, g) complete, $\mathrm{Ric} \geq \frac{n-1}{r^2} > 0$ $(\exists r > 0)$

$\Rightarrow$ M cpct and $\mathrm{diam}(M) \leq \pi r$

Ex sphere of radius r



$\hookrightarrow$ $\mathrm{Ric} = \frac{n-1}{r^2}$

and $\mathrm{diam} = \pi r$

Cor: if M cpct, $Ric > 0$ $\quad (\Rightarrow Ric \geq a > 0)$

$\Rightarrow \pi_1 M$ finite

proof of cor: $\tilde{M}$ = universal cover, $\tilde{g} = p^* g$ (complete)

$\tilde{M} \downarrow p$ M

and $\widetilde{Ric} \geq a > 0$

$\Rightarrow \tilde{M}$ cpct

$\Rightarrow p^{-1}(x)$ = finite $= \# \pi_1(M)$ $\square$

proof of thm: ETS every minz geodesic has length $\leq \pi r$

let $\gamma : [0, \ell] \to M$ = minz geodesic. PBAL (Hopf-Rinow)

E_i = parallel ON basis along γ, $E_n = \gamma'(t)$

consider proper variations $V_i(t) = \sin\left(\frac{\pi t}{\ell}\right) E_i(t)$ $\quad i = 1, \dots, n-1$

↳ notice $V_i = V_i^\perp$

$\left(\text{variation } \gamma_s(t) = \exp_{\gamma(t)}(s V_i(t)) \right)$

↳ $\left.\frac{d^2}{ds^2}\right|_{s=0} L\gamma_s = I_\ell(V_i^\perp, V_i^\perp) = I_\ell(V_i, V_i)$

$$= \int_0^\ell \left| \frac{D}{dt} V_i \right|^2 - R(V_i, \gamma', \gamma', V_i) \, dt$$

$$= - \int_0^\ell \langle V_i, V_i'' \rangle + R(V_i, \gamma', \gamma', V_i) \, dt$$

$$= -\int_0^l \sin\left(\frac{\pi t}{l}\right)\left(-\left(\frac{\pi}{l}\right)^2 + R(E_i, \gamma', \gamma', E_i)\right) dt$$

$$\sum_{i=1}^{n-1} I_l(V_i, V_i) = -\int_0^l \sin\left(\frac{\pi t}{l}\right)\left(-(n-1)\left(\frac{\pi}{l}\right)^2 + \underbrace{\sum_{i=1}^{n} R(E_i, \gamma', \gamma', E_i)}_{Ric(\gamma', \gamma') \geq \frac{n-1}{r^2}}\right) dt$$

$$\underset{hyp}{\leq} -(n-1)\int_0^l \sin\left(\frac{\pi t}{l}\right)\left(-\left(\frac{\pi}{l}\right)^2 + \frac{1}{r^2}\right) dt$$

$$< 0 \quad \text{if} \quad -\left(\frac{\pi}{l}\right)^2 + \frac{1}{r^2} > 0$$

$$\iff l > \frac{\pi}{r}$$

so if $l > \frac{\pi}{r} \Rightarrow \exists i$ st. $I_l(V_i, V_i) < 0$

$$\overset{\shortparallel}{\frac{d^2}{ds^2} L\gamma_s} \Rightarrow \gamma \text{ not minz } \square$$

Weinstein - Synge: spse (M^n, g) cpct, $K > 0$ (sectional)

$F : M \to M$ isometry

assume: ① n even, F preserves orientation

② n odd, F reverses orientation

$\Rightarrow F$ has fixed pt

Cor: M as above

then: (1) n even and orientable $\Rightarrow \pi_1 M = \{0\}$

(2) n odd $\Rightarrow M$ = orientable

Note! if n even, then $\pi_1 M = \begin{cases} 0 & \text{if orientable} \\ \mathbb{Z}/_2 & \text{if not orientable} \end{cases}$

proof of cor: (1) $\begin{array}{c} \widetilde{M} \\ \downarrow p \\ M \end{array}$ = universal cover (= orientable)

$\hookrightarrow \widetilde{g} = p^* g$

since $K \geq \delta > 0 \Rightarrow \widetilde{K} \geq \delta > 0$

(Bonnet-Myers) $\Rightarrow \widetilde{M}$ cpct

given deck transformation $f : \widetilde{M} \to \widetilde{M}$

(= isometry) $\Rightarrow f$ has fixed pt.

$\Rightarrow f = id$

diffeo, $f(p^{-1}(x)) = p^{-1}(x)$

(since M orientable, $p : \widetilde{M} \to M$ preserves orientation

$\Rightarrow f$ preserve orientation)

(2) n odd, M not orientable

$\begin{array}{c} \overline{M} \\ \downarrow p \\ M \end{array}$ = dbl cover = oriented, cpct, $K \geq \delta > 0$

$(\bar{g} = p^* g)$

$\Rightarrow$ deck transformations $= \{id, \tau\}$

where τ : swaps orientation of $\overline{M}$

$\Rightarrow \tau$ has fixed pt $\Rightarrow \tau = id$ ↯ $\square$

lemma: if $A \in O(n-1)$ st. $\det A = (-1)^n$

$\Rightarrow \exists\, v \in \mathbb{R}^{n-1} \neq 0$ st $Av = v$

proof of lemma: $\exists$ ON basis $e_1 \cdots e_{n-1}$ st. in this basis

$$A = \begin{pmatrix} R_{\theta_1} & & & & & \\ & R_{\theta_2} & & & & \\ & & \ddots & & & \\ & & & \pm 1 & & \\ & & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}$$

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

block diagonal

$\det R_\theta = 1$

n even $\Rightarrow \det A = 1 \Rightarrow \dim\{\text{e'val} = -1\} = \text{even}$

but $n-1 = \text{odd}$

$\Rightarrow \exists$ e'vec with e'val $= 1$

n odd $\Rightarrow \det A = -1 \Rightarrow \dim\{\text{e'val} = -1\} = \text{odd}$

at ≥ 1

but $n-1 = \text{even}$

$\Rightarrow \exists$ e'vec with e'val $= 1$ $\square$

proof of thm: assume F has no fixed pt

$M \ni p \mapsto d(p, F(p)) > 0$ (and continuous)

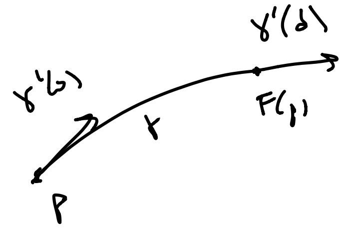
$\Rightarrow \exists\, p \in M$ st. $d(p, F(p)) = d > 0$

minimized

let $\gamma : [0, d] \to M$ min'z geodesic $p \to F(p)$

PBAL

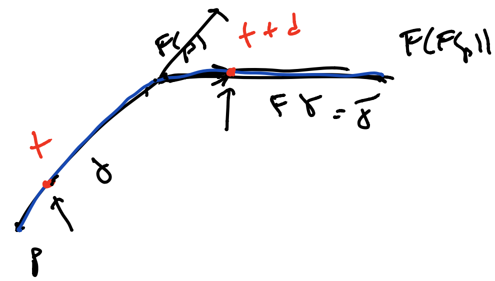
Claim: $DF|_p \gamma'(0) = \gamma'(d)$



$\tilde{\gamma}(t) = (F \circ \gamma)(t)$

$=$ geodesic $F(p) \to F(F(p))$

(minimizing)



WTS: composition

$\sigma: \; p \xrightarrow[\gamma]{} F(p) \xrightarrow[\tilde{\gamma}]{} F(F(p))$

minimizing

$d \leq d(F(\gamma(t)), \gamma(t)) \leq d$

since $d =$ least distance $p \to F(p)$

since $L\sigma|_{[t,t+d]} = d$

so $\sigma|_{[t,t+d]} =$ min'z $\quad \forall t \in [0,d]$

$\Rightarrow \sigma$ smooth $\Rightarrow \sigma'(d^-) = \sigma'(d^+)$

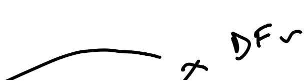
$\sigma'(d^-) = \gamma'(d)$

$\sigma'(d^+) = \frac{d}{dt}\Big|_0 (F \circ \gamma)(t) = DF|_p \gamma'(0) \qquad \square$

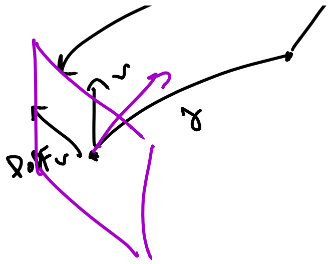
define $A: T_pM \to T_pM$

$A = P \circ DF|_p$

P: parallel transport along γ from $\gamma(d) = F(p)$ to $\gamma(0) = p$



$\Rightarrow A =$ isometry



and $A\gamma'(0)$

$= P \circ DF|_p \gamma'(0)$

$= P\gamma'(\ell)$ since γ geodesic

$= \gamma'(0)$

let $V = \underline{\gamma'(0)^\perp} \subset T_pM$

$\Rightarrow A : V \to V$ isometry

$V \cong \underline{\mathbb{R}^{n-1}}$ via ONB basis

$\left(\begin{array}{l} \text{if } v \perp \gamma'(0) \\ \text{then } \langle Av, \gamma'(0)\rangle \\ \quad = \langle Av, A\gamma'(0)\rangle \\ \quad = \langle v, \gamma'(0)\rangle = 0 \end{array} \right)$

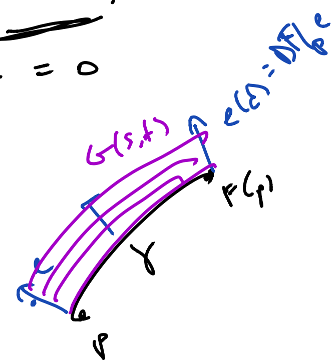
and $\det A = (-1)^n$ by hyp.

↳ $\det(A|_V) = \underline{(-1)^n} \Rightarrow \exists$ unit vector $e \in \gamma'(0)^\perp$ s.t. $\underline{Ae = e}$

let $e(t)$ = parallel transport of e along γ

define $G(s,t) = \exp_{\gamma(t)}(s\,e(t))$ $\quad$ $G(0,t) = \underline{\gamma(t)}$

= variation of curves s.t. $\underline{D_{\partial_s G}\partial_s G = 0}$



and $\underline{G(s,\ell)} = \exp_{\gamma(\ell)}(s\,e(\ell))$

$= \exp_{F(p)}(s\,DF|_p(e))$

↑ since $P \circ DF|_p e = e$

$DF|_p e = P^{-1}e$

$= \underline{e(\ell)}$

$= F(\exp_p(se))$

↑ since F isometry

$$= F(G(s,0))$$

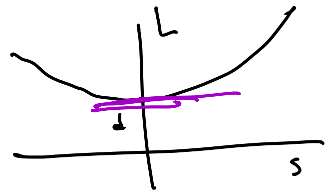
by defin of G

So $d(G(s,d), G(s,0)) = d(F(G(s,0)), G(s,0))$
$\quad\quad\geq d$ @ $\quad = d$ @ $s=0$

$\quad \leq$

$LG(s,\cdot)$

$$\Rightarrow \quad \frac{d^2}{ds^2}\Big|_{s=0} LG(s,\cdot) \geq 0$$



but
$$\frac{d^2}{ds^2}\Big|_{s=0} LG(s,\cdot) = \int_0^d \underbrace{\left|\frac{D}{dt}V^\perp\right|^2}_{0} - R(V^\perp,\gamma',\gamma',V^\perp)\,dt + \underbrace{\langle \gamma', \nabla_V V\rangle\Big|_0^d}_{0}$$

where $V = \partial_s G|_{s=0}$ $\quad$ since $\nabla_{\partial_s}\partial_s = 0$

$$V = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(t)}(s e(t))$$
$$= e(t) \Rightarrow V \perp \gamma', \ \frac{DV}{dt} = 0$$

$$= \int_0^d -R(e,\gamma',\gamma',e)\,dt$$

$$= \int_0^d -K|_{\gamma(t)}(e,\gamma')\,dt < 0$$ D

Conjugate pts and stability

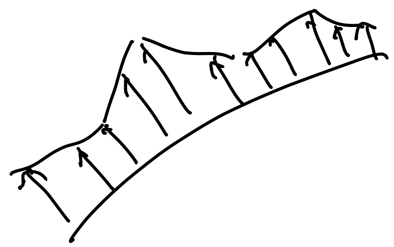
idea: γ = geodesic ($\Leftrightarrow$ stationary for length)

↳ γ stable for L until first conj. pt unstab for L after first conj. pt

piece-wise-smooth variation:

$\gamma : [0,\ell] \to M$ geodesic, PBAL

$V(t) : [0,\ell] \to T_{\gamma(t)}M$ continuous, p.w. smooth field on γ



i.e. $\exists\ 0 = t_0 < t_1 < \dots < t_k = \ell$

s.t. $V\big|_{(t_i,t_{i+1})}$ smooth, V cont.

$\Rightarrow \exists$ variation $F(s,t) : (-\varepsilon,\varepsilon)\times[0,\ell] \to M$

$$\gamma_s(t) = \underline{\exp_{\gamma(t)}(sV(t))} \qquad \text{"linear"}$$

s.t. F continuous, smooth in s, $\nabla_{\partial_s F}\partial_s F = 0$

p.w. smooth: $F\big|_{(-\varepsilon,\varepsilon)\times(t_i,t_{i+1})}$ = smooth

and $\partial_s F\big|_{s=0} = V(t)$

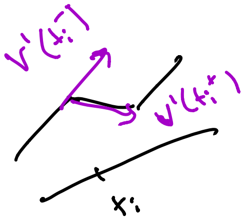
and if $V(t) = 0 \Rightarrow F(s,t) = F(s,0)$

↳ if $V(0) = V(\ell) = 0$ then F is "proper"

$$\hookrightarrow \frac{d^2}{ds^2} L\gamma_s = \underbrace{\int_0^\ell \left|\frac{D}{dt}V^\perp\right|^2 - R(V^\perp,\gamma',\gamma',V^\perp)\,dt}_{I_\ell(V^\perp,V^\perp)} \qquad (\text{since } \nabla_V V = 0)$$

$$= -\int_0^\ell \langle V^\perp, \mathcal{J}V^\perp\rangle\,dt - \sum_{i=1}^{k-1} \langle V^\perp(t_i), \frac{DV^\perp}{dt}(t_i^+) - \frac{DV^\perp}{dt}(t_i^-)\rangle$$

$$+ \langle V^\perp(\ell), \tfrac{DV^\perp}{dt}(\ell)\rangle - \langle V^\perp(0), \tfrac{DV^\perp}{dt}(0)\rangle$$



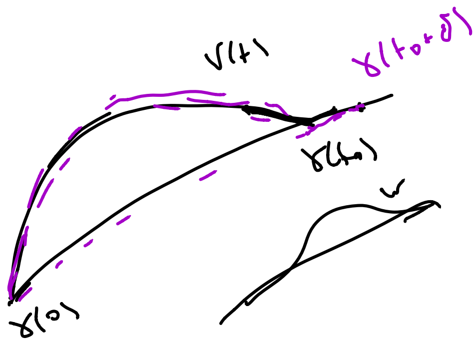
$$\mathcal{J}V = V'' + R(V,\gamma')\gamma'$$

(same computation)

<u>thm</u>: if $\gamma(t_0)$ conjugate to $\gamma(0)$ along γ

$\Rightarrow \gamma|_{[0,t_0+\delta]}$ <u>not minz</u> for any $\delta > 0$

<u>proof</u>:



$\exists$ non-zero Jacobi field $V : [0,t_0] \to TM$ on γ

st $V(0) = V(t_0) = 0$, $V \perp \gamma$ and $V' \perp \gamma$

$\hookrightarrow V'(t_0) \neq 0$

choose W any smooth field on γ

st $W \perp \gamma$, $W(0) = W(t_0+\delta) = 0$

and $W(t_0) = -V'(t_0)$

(extend V to $[0, t_0+\delta]$ by 0)

$\hookrightarrow V + \varepsilon W$ = p.w. field, vanishes @ 0, $t_0+\delta$

($\Rightarrow \exists$ "nice" proper variation γ_s realizing $V+\varepsilon W$)

$$\left[\, I(V,W) = \int \langle V',W'\rangle - R(V,\gamma',\gamma',W) \right.$$

$$\hookrightarrow \frac{d^2}{ds^2} L\gamma_s = I_{t_0+\delta}(V+\varepsilon W, V+\varepsilon W)$$

$$= \underline{I(V,V)} + 2\varepsilon \underline{I(V,W)} + \varepsilon^2 I(W,W)$$

$$= -\int_0^{t_0+\delta} \langle V, \cancel{\nabla\nabla} V\rangle^{0} + \langle \cancel{V(t_0)}^{0}, V'(t_0^-)\rangle$$

$$+ 2\varepsilon\left(-\int_0^{t_0+\delta} \langle W, \cancel{\nabla\nabla} V\rangle^{0} + \langle \underbrace{W(t_0)}_{= -V'(t_0^-)}, V'(t_0^-)\rangle \right)$$

$$+ \varepsilon^2 I(W,W)$$

$$= -2\underline{\varepsilon}\, |V'(t_0)|^2 + \underline{\varepsilon^2}\, I(W,W)$$

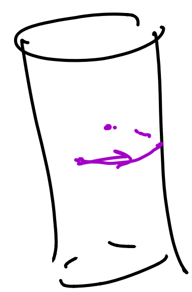
< 0 for ε small

and $\left.\frac{d}{ds}\right|_{s=0} L\gamma_s = 0$ since γ geodesic, variation proper $\square$

<u>Note</u>: converse not true $S^1 \times \mathbb{R}$

↳ can be stable but not minz

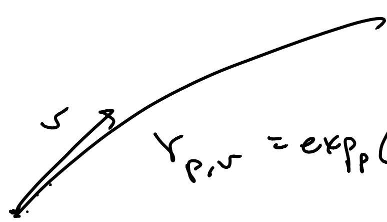
(b/c of topology)



cut locus, conjugate locus

(M,g) complete

$t_{cut}(v)$ = largest time $\leq \infty$ s.t. $\gamma_{p,v}\big|_{[0,t_{cut}]}$ minz



$\gamma_{p,v} = \exp_p(tv)$

$t_{conj}(v)$ = largest time $\leq \infty$ s.t. $\gamma\big|$

$|_{[0, t_{conj})}$ has no conj. pts to p

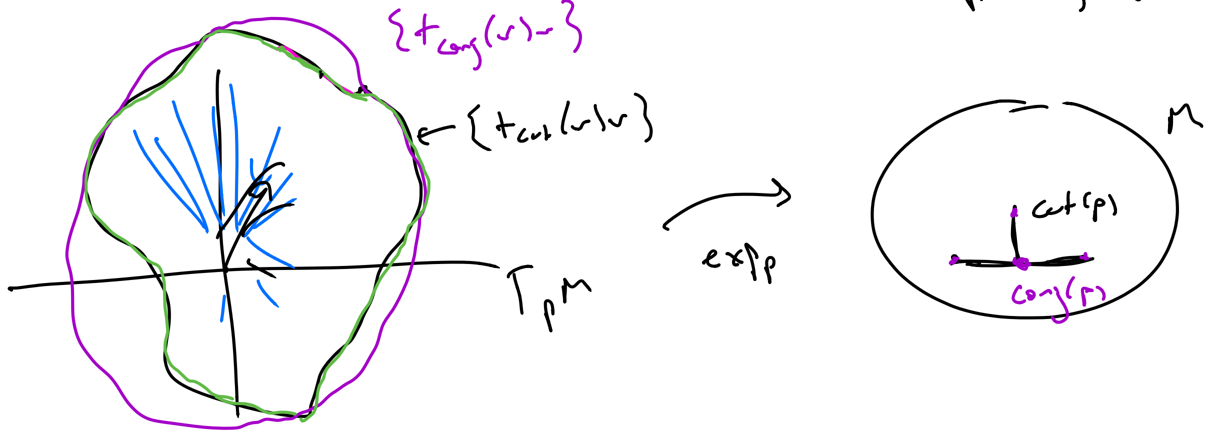
(thm $\Rightarrow t_{cut}(v) \le t_{conj}(v)$)

cut locus $= \{ q = \exp_p(t_{cut}(v) v) : v \in T_pM, |v| = 1 \}$

$cut(p) = \{ q : \exists$ minz $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$ s.t. $\gamma|_{[0,t]}$ not minz $\forall t > 1 \}$

$conj(p) = \{ \exp_p(t_{conj}(v) v) : v \in T_pM, |v| = 1 \}$

$= \{ q : \exists$ minz $\gamma : p \to q$ s.t. q = first conj. pt to p along $\gamma \}$



defin $\quad seg(p) = \{ t_{cut}(v) v : v \in T_pM \}$

$seg^0(p) = \{ tv : 0 \le t < t_{cut}(v), v \in T_pM \}$

So: $\exp_p(seg(p)) = cut(p)$

and $seg^0(p)$ = star-shaped $\supseteq B_1$

Note: $q \in M \Rightarrow \exists$ minz $\gamma(t) = \exp_p(tv) : p \to q$, $t \in [0,1]$

① if $\gamma|_{[0,1+\delta]}$ minz $\Rightarrow t_{cut}(v) > 1$

$\Rightarrow v \in seg^{\circ}(p)$ $(\Rightarrow q \notin cut(p))$

② else $\Rightarrow t_{cut}(v) = 1$

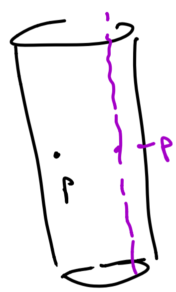
$\Rightarrow v \in seg(p)$ (or $q \in cut(p)$)

$\underline{\text{So}}$ $M \setminus cut(p) = \exp_p(seg^{\circ}(p))$

$\cong B_1$

"all topology captured in cut locus"

$\underline{\text{Ex}}$: $S^1 \times \mathbb{R}$



$cut(p) = \{-p\} \times \mathbb{R}$

$conj(p) = \emptyset$

S^2

$cut(p) = \{-p\}$

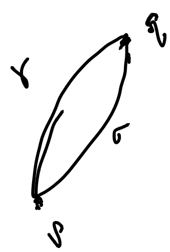
$= conj(p)$

$\underline{\text{lemma}}$: let $\gamma(t) = \exp_p(tv)$ geodesic PBAL $\gamma(0) = p$

$t_0 = t_{cut}(v)$

i.e. $\gamma(t_0) \in cut(p)$

then either: Ⓐ $\gamma(t_0) =$ (first) conj. pt to p along γ

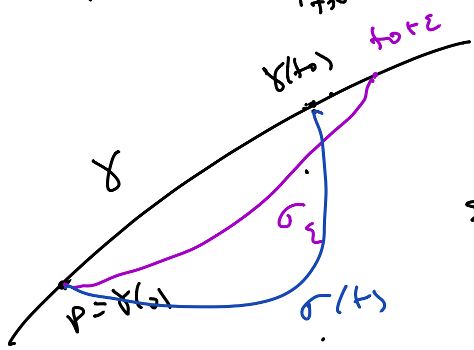


and/or (B) $\exists$ geodesic $\sigma = \exp_p(tw) \neq \gamma$.
st $\sigma(t_0) = \gamma(t_0)$, $L\sigma = L\gamma$

conversely, if (A) or (B) occur @ $t_0 > 0$
$\Rightarrow t_0 \geq t_{cut}(v)$

proof: ((A) $\Leftrightarrow$ $D\exp_p\big|_{t_0 v}$ singular) $\qquad \gamma(t) = \exp_p(tv)$

spse $D\exp_p\big|_{t_0 v}$ non-singular



let $\sigma_\varepsilon = \exp_p(t w_\varepsilon)$ PB/TL
$=$ min'z geodesic $p \to \gamma(t_0+\varepsilon)$

since $\gamma\big|_{[0,t_0+\varepsilon]}$ not min'z
$\hookrightarrow$ w_ε not parallel to v

$\exists\ \varepsilon_i \to 0$ st $v_{\varepsilon_i} \to w \in T_pM$
$\hookrightarrow$ $\sigma_{\varepsilon_i}(t) \to \sigma(t) = \exp_p(tw)$

$\Rightarrow$ $\sigma\big|_{[0,t_0]}$ = min'z geodesic $p \to \gamma(t_0)$

$\left(L\sigma\big|_{[0,t_0]} = t_0 = L\gamma\big|_{[0,t_0]} = d(p,\gamma(t_0)) \right)$

if $\overset{\sigma'(0)}{\overset{\shortparallel}{w}} \neq \overset{\gamma'(0)}{\overset{\shortparallel}{v}}$ $\Rightarrow$ B occurs

if $w = v$ $\Rightarrow$ $\exp_p(\underbrace{(t_0+\varepsilon_i)v}_{t_0 v = t_0 w}) = \gamma(t_0+\varepsilon_i)$ length $L\sigma_i$
$= \exp_p(\underbrace{l_i}_{} w_{\varepsilon_i})$

$\swarrow$ $t_0 w$

but $D\exp_p|_{t_0 v}$ non-sing.

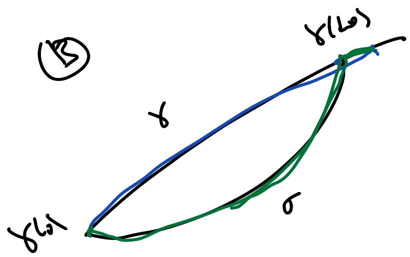
$\Rightarrow \exp_p$ loc. diffeo near $t_0 v$

$\Rightarrow (t_0+\varepsilon_i)v = l_i w_{\varepsilon_i}$

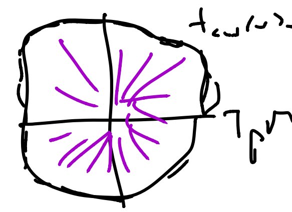
$\Rightarrow w_{\varepsilon_i}$ parallel to v $\lightning$

if Ⓐ ($\gamma(t_0)$ conj. to $\gamma(0)$) $\Rightarrow \gamma|_{[0,t_0+\delta]}$ not min'z

Ⓑ



$\Rightarrow$ ditto



$\square$

Cor: ① $q \in cut(p) \Leftrightarrow p \in cut(q)$

② $v \mapsto t_{cut}(v)$ continuous
$\in T_pM \cap \{|v|=1\}$

③ $\{tv : v \in T_pM,\ t < t_{cut}(v)\} = seg^0(p)$ open + star-shaped

and $\exp_p : seg^0(p) \to M \setminus cut(p)$ = diffeo

and $cut(p)$ = closed

and $dist(\cdot,p)^2$ = smooth on $M\setminus cut(p)$

proof: ① Ⓐ, Ⓑ both symmetric

② take $v_i \to v$ unit vectors in T_pM

WTS: $t_c(v_i) \to t_c(v)$ (possibly $=\infty$)

wlog spse $t_c(v_i) \to T \in (0,\infty]$

↳ WTS: $T = t_c(v)$

on one hand: $\gamma_i(t) = \exp_p(tv_i)$, $\gamma_i|_{[0,t_c(v_i)]}$ = minz

$\downarrow$

$\gamma(t) = \exp_p(tv)$ $\Rightarrow \gamma|_{[0,T]}$ = minz

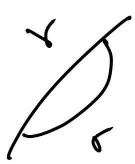
$\Rightarrow t_c(v) \geq T$

OTOH: either Ⓐ, or Ⓑ occurs $\forall i$

↳ Ⓐ: $D\exp_p|_{t_c(v_i)v_i}$ singular $\forall i$

$\Rightarrow D\exp_p|_{Tv}$ singular $\Rightarrow t_c(v) \leq T$

↳ Ⓑ not Ⓐ: $\exists$ $\overset{\text{unit vector}}{w_i \neq v_i}$ st. $\left.\begin{aligned}&\exp_p(t_c(v_i)v_i)\\ &= \exp_p(t_c(v_i)w_i)\end{aligned}\right\}$



wlog $w_i \to w$

$\downarrow$

$\exp_p(Tw) = \exp_p(Tv)$

if $w \neq v$ $\Rightarrow$ Ⓑ occurs in limit

$\Rightarrow t_c(v) \leq T$

if $w = v$ (and by assumption $D\exp_p|_{Tv}$ non-sing)

||

" $\exp_p$ = diffeo near Tv

$\Rightarrow t_c(v_i)v_i = t_c(w_i)w_i \quad \forall i \gg 1$

$\Rightarrow v_i = w_i$ ↯

③ $\mathrm{seg}^0(p) = \{tv : v \in T_pM,\ t < t_{cut}(v)\}$ = open

since $t_{cut}(v)$ continuous

by lemma, $D\exp_p|_{tv}$ = non-singular for $tv \in \mathrm{seg}^0(p)$

$\Rightarrow \exp_p$ loc. diffeo $\mathrm{seg}^0(p) \to M \setminus \mathrm{cut}(p)$

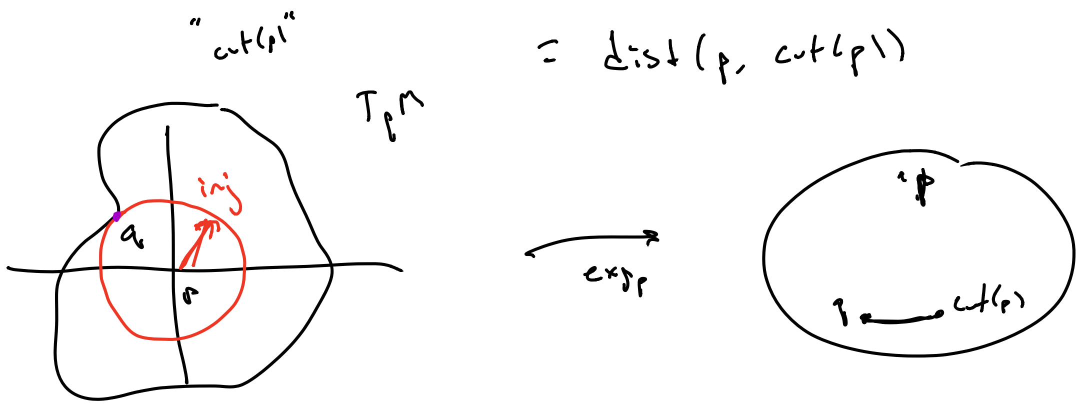
and $\exp_p|_{\mathrm{seg}^0(p)}$ injective

and $\mathrm{dist}(x,p) = |\exp_p^{-1}(x)|$ if $x \in M \setminus \mathrm{cut}(p)$ etc.. □

injectivity radius @ p = largest R s.t. $\exp_p|_{B_R(0)}$

= diffeo onto image

= $\mathrm{dist}(p, \mathrm{cut}(p))$



prop (Klingenberg): $q \in \mathrm{cut}(p)$, $d(p,q) = d(p, \mathrm{cut}(p))$

then either: ① $q \in$ conjugate locus of p

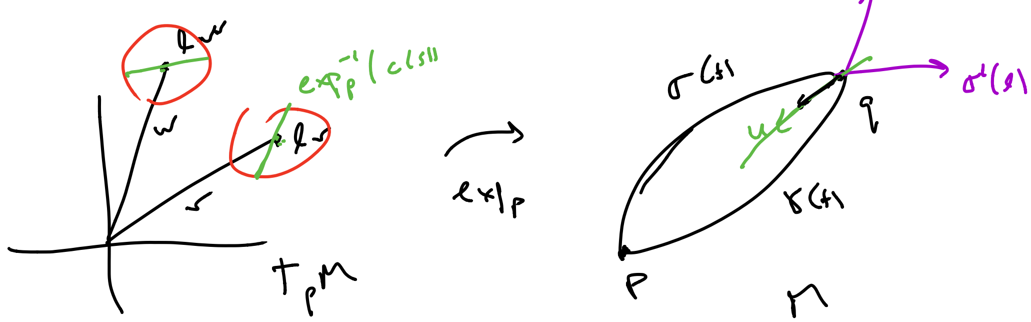
(i.e. $\exists$ min'z geodesic $\gamma : p \to q$
s.t. q = conj. to p)

② $\exists$ exactly 2 min'z geodesics
$\gamma, \sigma : p \to q$ $\quad L\gamma = L\sigma = \ell$

and $\gamma'(\ell) = -\sigma'(\ell)$

proof: spse $q \notin$ conj. locus of p

(lemma) $\Rightarrow$ $\exists\ \sigma \neq \gamma$ min'z geodesics $p \to q$
$\gamma(t) = \exp_p(tv)$, $\sigma(t) = \exp_p(tw)$ $\quad |v| = |w| = 1$
$t \in [0, \ell]$



by assumption, $D\exp_p|_{\ell w}$, $D\exp_p|_{\ell v}$ non-singular

$\Rightarrow$ $\exp_p$ loc. diffeo near ℓv, ℓw

spse $\gamma'(\ell) \neq -\sigma'(\ell)$

$\Rightarrow$ $\exists\ u \in T_qM$ st. $\langle u, \gamma'(\ell)\rangle < 0$
$\langle u, \sigma'(\ell)\rangle < 0$

pick any curve $c(s) : (-\varepsilon, \varepsilon) \to M$ st. $c(0) = q$
$c'(0) = u$

$\Rightarrow \exists$ curves $c_1, c_2 : (-\varepsilon, \varepsilon) \longrightarrow T_pM$

s.t. $\ell c_1(0) = \ell v$ $\quad c(s) = \exp_p(\ell c_i(s)) \quad i=1,2$

$\ell c_2(0) = \ell w$

define $\gamma_s(t) = \exp_p(t c_1(s))$, $\sigma_s(t) = \exp_p(t c_2(s))$

$\hookrightarrow \underline{\gamma_s(0)} = \sigma_s(0) = p$, $\underline{\gamma_s(\ell)} = \sigma_s(\ell) = c(s)$

now $\displaystyle \frac{d}{ds}\Big|_{s=0} L\gamma_s = \int_0^\ell \langle \partial_s \gamma_s\big|_{s=0}, \cancel{\frac{D\gamma'}{dt}} \rangle \, dt + \langle \partial_s\big|_{s=0}, \gamma'_s \rangle \Big|_0^\ell$

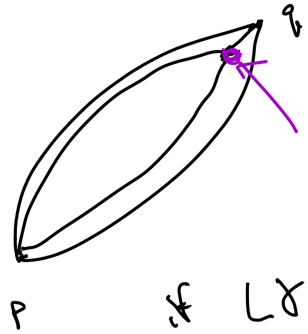
0

$= \langle c'(\ell), \gamma'(\ell) \rangle = \langle u, \gamma'(\ell) \rangle < 0$

b/c $\displaystyle \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s(0) = 0$, $\displaystyle \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s(\ell) = \frac{\partial}{\partial s} c(s) = c'(s)$

ditto for σ_s $\quad \Rightarrow L\gamma_s < L\gamma$ and $L\sigma_s < L\sigma$

for $s > 0$ small



$\in cut(p)$, closer to p than q $\lightning$

if $L\gamma_s = L\sigma_s \Rightarrow c(s) \in cut(p)$

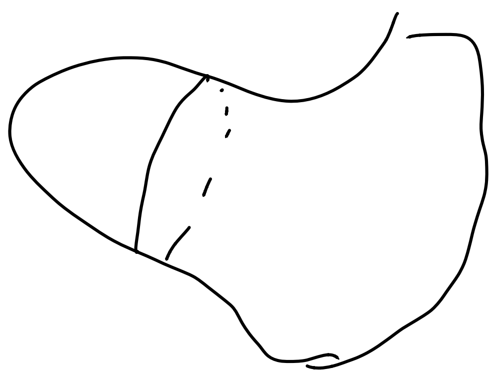
if $L\gamma_s < L\sigma_s \Rightarrow \sigma_s$ not minz past ℓ

$\Rightarrow c(s) \in cut(p)$

but $d(c(s), p) \leq L\gamma_s < \ell$ $\quad \lightning \quad \square$

in second case, γ, σ connect up to form smooth geodesic loop

idea: $\mathrm{inj}(p)$ bounded below by $\min \begin{cases} \text{dist to first conj. pt} \\ \frac{1}{2} \text{ length of any geodesic loop} \end{cases}$



idea: Jacobi field on const curvature k $= w\, \mathrm{sn}_k(t)$

J st $J(0) = 0$, $J'(0) = w$

$$= \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, t) & k > 0 \\ t & k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, t) & k < 0 \end{cases}$$

upper bound on curvature

$\rightsquigarrow$ lower bound on dist to conj. pts

lemma: let (M,g) sectional curvature $K \le k$ for $k \ge 0$

$\gamma : [0,\ell) \longrightarrow M$

$V(t)$ = Jacobi field on γ, $V(0) = 0$, $V'(0) = w$

$\Rightarrow |V(t)| \ge |w| \sin_k(t)$

propn: (M,g) complete, $K \le k$ for $k \ge 0$

$\Rightarrow \operatorname{inj}(p) \ge \min\left(\frac{\pi}{\sqrt{k}},\ \frac{1}{2} \text{ length of smallest geodesic loop} \right)$

proof of propn: since $\operatorname{cut}(p)$ closed

$\Rightarrow \exists\, q \in \operatorname{cut}(p)$ s.t. $d(p,q) = d(p, \operatorname{cut}(p))$
$= \operatorname{inj}(p)$

(propn above) $\Rightarrow$ either q first conj. pt to p along some minz geodesic

or $d(p,q) = \frac{1}{2}$ length of some geodesic loop

but $\sin_k(t) > 0$

for $0 < \sqrt{k}\, t < \pi$

$\Rightarrow d(p,q) \ge \frac{\pi}{\sqrt{k}}$ $\square$

$i(M) = \inf_{p \in M} \operatorname{inj}(p)$

= "injectivity radius of M"

if $i(M)$ realized by $p, q \in cut(p)$

$\Rightarrow$ either q conj. to p along some minz geodesic

or $\exists$ minz $\gamma \neq \sigma : p \to q$

s.t. γ, σ close up to form smooth geodesic loop

[$\Rightarrow p \in cut(q)$ and $d(p,q) = i(M)$

$\Rightarrow$ apply Klingenberg's thm to both p, q]

<u>Note:</u> $t_{cut}(p,v)$ = first time geodesic $\gamma_{pv}(t) = \exp_p(tv)$ fails to be minz

$: TM_1 = \{(p,v) : |v| = 1\} \longrightarrow (0, \infty]$ continuous

if M cpct (w/o ∂M), then $i(M)$ realized

$\Rightarrow i(M) \geq \min\left(\underline{\text{dist between conj pts}}, \ \frac{1}{2} \text{length of any geodesic loop} \right)$

$\geq \min\left(\frac{\pi}{\sqrt{k}}, \ \frac{1}{2} \text{length} \ldots \right)$

if $K \leq k$

<u>Note:</u> (Cartan) can realize free homotopy classes by geodesic loops

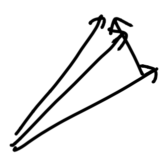
Comparison Geometry

"bounds on curvature $\rightsquigarrow$ bounds on geometry"

$V(t) = \underset{\sim}{D}\exp_p\big|_{tv}(tw)$ = Jacobi field along $\gamma(t) = \exp_p(tv)$

$V(0) = 0, \quad V'(0) = w$

(variation $\exp_p(t(v+sw))$)

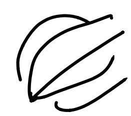


$\longrightarrow$ $K = k = \text{const}$, $|v| = 1$, $w \perp v$

then $V(t) = w \, \sin_k(t)$, $\quad \sin_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, t) & k > 0 \\ t & k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, t) & k < 0 \end{cases}$

$\longrightarrow$ K "more positive"

metric "more scrunched"



$\updownarrow$ $\quad$ since $|D\exp_p|_v(w)| = |V(1)|$

"$|V|$ smaller"

"easy comparison": (M, g), $\underset{\text{sect curv}}{K} \le k$, $\gamma : [0, \ell] \to M$ geodesic PBAL

$V(t)$ = Jacobi field on γ

$V(0) = 0, \; V'(0) = w \perp \gamma'$

$\Rightarrow |V(t)| \ge |w| \sin_k(t)$

Cor: (M as above), in normal coords (x^i), $w \perp X$

$$\Rightarrow g|_x(w,w) \geq |w|^2_{eucl.} \frac{\sin_k^2(|x|)}{|x|^2}$$

$\underline{\text{Cor}}$: - - - - , dist between conjugate pts $\geq \frac{\pi}{\sqrt{k}}$

<u>proof of comparison thm:</u>

note that $\underline{|V(t)|} \neq 0$ for $t \in (0,\delta]$

Jacobi eqn:

$\underline{V'' + R(V,\gamma')\gamma' = 0}$

(scalar eqn if $n=2$)

$$\frac{d^2}{dt^2}|V(t)| = \frac{d}{dt}\frac{\langle V, V'\rangle}{|V|}$$

$$= \frac{\langle V, V''\rangle}{|V|} + \underbrace{\frac{\langle V', V'\rangle}{|V|} - \frac{\langle V, V'\rangle^2}{|V|^3}}$$

$$= \frac{1}{|V|}\Big(\underbrace{-R(V,\gamma',\gamma',V)}_{= -K(\gamma',V)|V|^2|\gamma'|^2,\ |\gamma'|=1} + \underbrace{\Big|V' - \frac{\langle V,V'\rangle V}{|V|^2}\Big|^2}_{\geq 0}\Big)$$

$(= 0$ if $n=2)$

$$\geq -k|V|$$

let $u = |V(t)|$, $\quad v = \underline{|w|\sin_k(t)}$

then $\underline{u'' + ku \geq 0}$, $\quad u(0) = 0$, $\quad \underline{u'(t) \to |w|}$ as $t \to 0$

$v'' + kv = 0$, $\quad v(0) = 0$, $\quad \underline{v'(0)} = |w|$

$$V(t) = tw + O(t^2) \Rightarrow u' = \frac{\langle V, V'\rangle}{|V|} = \frac{\langle tw + O(t^2), w + O(t)\rangle}{t|w| + O(t^2)}$$

$$\to |w|$$

$F = \underline{u'v} - uv' = 0$ @ 0

$F' = \underline{u''v} + \cancel{u'v'} - \cancel{u'v'} - uv'' \geq -kuv + kuv = 0$

so $F \geq 0$ (1'' elliptical)

$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \geq 0$ and $\frac{u}{v} \to 1$ as $t \to 0$

so $\frac{u}{v} \geq 1 \Rightarrow |V| \geq |w| \sin_k(t)$ $\square$

Q: what if $K \geq k$ const? OR M and $\tilde{M}$ s.t. $K \leq \tilde{K}$?

↳ still have comparison, but more complicated...

Rmk: if $n=2$ then same proof works for more general comparison

index inequality

$\gamma : [0,\ell] \to M$ geodesic PBAL, $t_0 \leq \ell$

index form $I_{t_0}(V,W) = \int_0^{t_0} \langle \frac{DV}{dt}, \frac{DW}{dt} \rangle - R(V,\gamma',\gamma',W)\, dt$

↑ V, W vector fields defined on γ

symmetric, bilinear

recall: if $\gamma_s(t)$ = variation of curves, $\gamma_0 = \gamma$, $V = \frac{\partial \gamma_s}{\partial s}\Big|_{s=0}$

$\Rightarrow \frac{d^2}{ds^2}\Big|_{s=0} L\gamma_s = I_\ell(V^\perp, V^\perp) + \langle \nabla_V V, \gamma' \rangle \Big|_0^\ell$, $V^\perp = V - \langle V,\gamma'\rangle\gamma'$

$$I_{t_0}(V,W) = -\int_0^{t_0} \langle V, \mathcal{L}W\rangle \, dt + \langle V, W'\rangle \Big|_0^{t_0}$$

where $\underline{\mathcal{L}W} = W'' + R(W,\gamma')\gamma'$

so $\mathcal{L}V = f$

$$\iff \underline{I_{t_0}(V,W)} = -\int_0^{t_0} \langle f, V\rangle \, dt \quad \forall \text{ vector fields } W \text{ vanishing @ } 0, t_0$$

need to understand the analysis of $\mathcal{L} \iff \underline{I_{t_0}}$

for $0 < t_0 \le l$

define $\lambda(t_0) = \inf \left\{ \underline{I_{t_0}(V,V)} : V \text{ p.w. smooth vector field on } \gamma,\ V \perp \gamma,\ V(0) = V(t_0) = 0,\ \underline{\int_0^{t_0} |V|^2 = 1} \right\}$

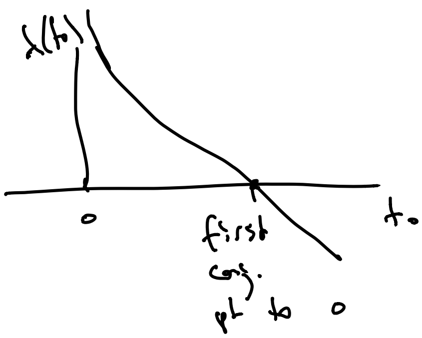
"= first eigenvalue of $\mathcal{L}$"

__thm__: ① $\lambda(t_0)$ finite, realized by a smooth soln to

$$\begin{cases} \mathcal{L}V + \lambda(t_0) V = 0 \\ V(0) = V(t_0) = 0 \end{cases} \qquad \mathcal{L}V = V'' + R(V,\gamma')\gamma'$$

② $\lambda(t_0)$ strictly decreasing in t_0, continuous, $\lambda(t_0) > 0$ for small t_0

③ $\lambda(t_0) > 0 \iff \nexists$ conj. pts in $[0, t_0]$



proof (sketch): ① define "$W^{1,2}$-norm" on V

$$\|V\|^2_{W^{1,2}[0,t]} = \int_0^t |V|^2 + \int_0^t |V'|^2$$

"Sobolev space" $W^{1,2}_0[0,t]$ = completion of $\left\{ \begin{array}{l} \text{smooth}, \perp \text{ vector fields} \\ \text{on } \gamma \text{ s.t. } V(0) = V(t) = 0 \end{array} \right\}$

$\Downarrow$

$V \iff V(0) = V(t) = 0, \quad V \perp \gamma$

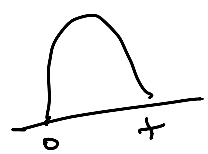
and $V' \in L^2$

$$|I_t(V,V)| = \left| \int_0^t |V'|^2 - R(V,\gamma',\gamma',V) \right| \le C \|V\|^2_{W^{1,2}[0,t]}$$

$$\Rightarrow \lambda(t) = \inf \left\{ I_t(V,V) : V \in W^{1,2}_0[0,t], \int_0^t |V|^2 = 1 \right\}$$

② "control over $\lambda(t)$"

claim: $\lambda(t) \ge \dfrac{2}{t^2} - \sup\limits_{\gamma[0,t]} |K|$



FTC: $|V(t)|^2 \le \left(\int_0^t |V'| \, ds \right)^2 \le t \int_0^t |V'|^2$

$$\Rightarrow \underbrace{\int_0^t |V|^2}_{=1} \le \frac{t^2}{2} \int_0^t |V'|^2$$

$$\Rightarrow I_t(V,V) \ge \frac{2}{t^2} - \sup |K| \underbrace{\int_0^t |V|^2}_{=1}$$

fix t_0

to prove this: take seq $V_i \in W^{1,2}_0[0,t_0]$

s.t. $\int_0^{t_0} |V_i|^2 = 1, \quad I_{t_0}(V_i,V_i) \longrightarrow \lambda(t_0)$

$\hookrightarrow \int_0^{t_0} |V_i'|^2 = I_{t_0}(V_i,V_i) + \int_0^{t_0} R(V_i,\gamma',\gamma',V_i)$

$$\leq 2\lambda(t_0) + \sup_{\gamma(0,t)} |K| \underbrace{\int_0^{t_0} |V_i|^2}_{=1}$$

so $\|V_i\|_{W^{1,2}[0,t_0]} \leq C$ ind of i

$\Rightarrow$ (subseq) $V_i \longrightarrow V \in W_0^{1,2}[0,t]$

strongly in L^2 $\longrightarrow \int_0^{t_0} |V|^2 = 1$

"weakly" in $W^{1,2}$

$\Rightarrow \lambda(t_0) \leq I_{t_0}(V,V) \leq \liminf_i I_{t_0}(V_i,V_i) = \lambda(t_0)$

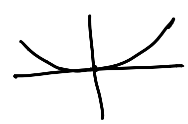
$\Rightarrow$ so $\lambda(t_0) = I_{t_0}(V,V)$

<u>WTS:</u> V smooth, solves $\mathcal{J}V + \lambda V = 0$

$\hookrightarrow$ use "stationarity"

take $W \in W_0^{1,2}[0,t_0]$, define $V_s = V + sW \in W_0^{1,2}[0,t_0]$

$\Rightarrow I_t(V_s,V_s) - \lambda(t_0)\int_0^{t_0} |V_s|^2 \geq 0$ and $= 0$ @ $s=0$



$$\Rightarrow 0 = \frac{d}{ds}\Big|_{s=0}(\cdots)$$

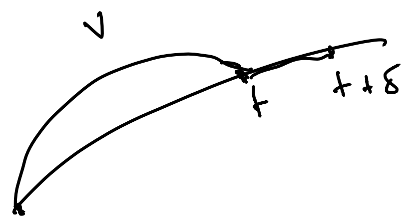
$$= 2\int_0^{t_0} \langle V', W' \rangle - R(V,\gamma',\gamma',W) - \lambda(t)\langle V, W\rangle \, dt \quad \forall W \in W_0^{1,2}$$

$\Rightarrow V$ "weak soln" to $\underline{\mathcal{J}V + \lambda V = 0}$

$\Rightarrow V$ smooth, classical soln to $\mathcal{J}V + \lambda(t_0)V = 0$

$(V(0) = V(t_0) = 0)$

since $V \in W_0^{1,2}[0,t_0]$



$V \in W_0^{1,2}[0,t] \Rightarrow V \in W_0^{1,2}[0,t+\delta]$

and $\int_0^{t+\delta} |V|^2 = \int_0^t |V|^2$

$\Rightarrow \lambda(t+\delta) \le \lambda(t)$

if $\mathcal{I}V + \lambda(t)V = 0$, $V(0)=V(t)=0$

$\Rightarrow V'(t) \ne 0$ (by uniqueness of ODEs)

$\Rightarrow V : [0,t+\delta] \to TM$ not smooth

$\Rightarrow$ cannot be soln to $\mathcal{I}V + \lambda(t)V = 0$, $V(0)=0$, $V(t+\delta)=0$

$\Rightarrow \lambda(t) \ne \lambda(t+\delta) \Rightarrow$ strictly decreasing

$$I_{t-\delta}(V,V) = \int_0^{t-\delta} |V'|^2 - R(V,\gamma',\gamma',V)\,dt, \quad \int_0^{t-\delta} |V|^2 \xrightarrow[\delta\to 0]{} \int_0^t |V|^2$$

$$\xrightarrow[\delta\to 0]{} I_t(V,V) = \lambda(t) \quad \text{by dominated convergence}$$

$$\lambda(t) \le \lambda(t-\delta) \le \frac{I_{t-\delta}(V,V)}{\int_0^{t-\delta} |V|^2\,dt} \le \lambda(t) + \varepsilon$$

$\varepsilon \to 0$ as $\delta \to 0$

$\Rightarrow \lambda(t)$ continuous

prove ①: t conj. to $0 \Leftarrow \exists$ non-zero Jacobi field V

$\mathcal{I}V = 0$, $V(0)=V(t)=0$

$\Rightarrow \lambda(t) \le 0$

if $\lambda(t) \le 0 \Rightarrow \exists\, t_*$ st $\lambda(t_*) = 0$

$\Rightarrow \exists\, V$ st $\mathcal{I}V + 0 = 0$, $V(0)=V(t_*)=0$

$\Rightarrow$ to conj. to 0 $\square$

<u>Rmk:</u> # neg e'vals = # conj. pts on γ to 0
(w/ multiplicity)

<u>index inequality</u>: $\gamma : [0,l] \to M$ geodesic PBAL, no conj. pts

V = smooth v.f. on γ, $\perp \gamma$

J = Jacobi field solving $J(0) = V(0)$, $J(l) = V(l)$
(exists since no conj pts)

<u>then</u>: $I_l(V,V) \geq I_l(J,J)$

<u>proof</u>: look @ $J - V$ $= 0$ @ $t = 0, l$

$\Rightarrow I_l(J-V, J-V) \geq 0$ since $\lambda(t) \geq 0$

$$\begin{gathered} I_l(J-V,J-V) \\ \shortparallel \\ I_l(V,V) - I_l(J,J) + \underbrace{2 I_l(J, J-V)}_{\shortparallel} \end{gathered} \quad \square$$

$$-2\int_0^l \langle J - V, \cancel{\nabla\nabla J}\rangle \, dt + 2\langle J', \cancel{J - V}\rangle \Big|_0^l$$

<u>index form comparison</u>: M^n, $\tilde{M}^n$

$\gamma : [0,l] \to M$
$\tilde{\gamma} : [0,l] \to \tilde{M}$ geodesics PBAL
assume $\tilde{\gamma}$ has no conj. pts

spse: $K_{\gamma(t)}(\gamma', X) \leq \tilde{K}_{\tilde{\gamma}(t)}(\tilde{\gamma}', \tilde{X})$
$\forall t$, $\forall X \perp \gamma'(t)$
$\forall \tilde{X} \perp \tilde{\gamma}'(t)$

take $V(t)$ smooth v.f. on γ, $V \perp \gamma$, $V(0) = 0$

$\tilde{J}(t)$ = any Jacobi field on $\tilde{\gamma}$
s.t. $\tilde{J} \perp \tilde{\gamma}$, $\tilde{J}(0) = 0$, $|\tilde{J}(l)| = |V(l)|$

then $I_l(V,V) \geq \tilde{I}_l(\tilde{J},\tilde{J})$

<u>proof:</u> $E_i(t)$, $\tilde{E}_i(t)$ = parallel orth bases along γ, $\tilde{\gamma}$

wlog $E_n = \gamma'$, $\tilde{E}_n = \tilde{\gamma}'$

if $V(l) = 0 \Rightarrow \tilde{J} \equiv 0$

else, wlog let $E_1 = \frac{V(l)}{|V(l)|}$, $\tilde{E}_1 = \frac{\tilde{J}(l)}{|\tilde{J}(l)|}$ @ $t = l$

$\hookrightarrow$ $V(t) = \sum_{i=1}^{n-1} a_i(t) E_i(t)$, define $\tilde{V} = \sum_{i=1}^{n-1} a_i(t) \tilde{E}_i(t)$

<u>notice:</u> $\tilde{V}(l) = \tilde{J}(l)$, $|V(t)|^2 = \sum a_i^2 = |\tilde{V}(t)|^2$

(and $\tilde{V}(0) = 0$) and $\tilde{V} \perp \tilde{\gamma}$

$$\rightarrow I_l(V,V) = \int_0^l \sum_i \left(\frac{da_i}{dt}\right)^2 - |V|^2 K\Big|_{\gamma(t)}(\gamma', V)\, dt \qquad \text{since } V \perp \gamma$$

$$\geq \int_0^l \sum_i \left(\frac{da_i}{dt}\right)^2 - |\tilde{V}|^2 \tilde{K}(\tilde{\gamma}', \tilde{V})\, dt$$

$$= \tilde{I}_l(\tilde{V}, \tilde{V}) \geq \tilde{I}_l(\tilde{J}, \tilde{J}) \qquad \square$$

<u>Rauch comparison thm:</u> $M^n, \tilde{M}^n$, $\gamma: [0,l] \to M$ PBAL
$\tilde{\gamma}: [0,l] \to \tilde{M}$

spse $\tilde{\gamma}$ has no conj. pts, $K\Big|_{\gamma(t)}(\gamma', X) \leq \tilde{K}\Big|_{\tilde{\gamma}(t)}(\tilde{\gamma}', \tilde{X})$ $\forall X \perp \gamma'$, $\forall \tilde{X} \perp \tilde{\gamma}'$

let $J(t)$, $\tilde{J}(t)$ Jacobi fields on γ, $\tilde{\gamma}$

st $J \perp \gamma$, $\tilde{J} \perp \tilde{\gamma}$, $J(0) = \tilde{J}(0)$, $|J'(0)| = |\tilde{J}'(0)|$

then $|J(t)| \geq |\tilde{J}(t)|$

proof: observe: $\frac{1}{2}\frac{d^2}{dt^2}|J|^2 = \frac{d}{dt}\langle J, J'\rangle$

$$= |J'|^2 - R(J,\gamma',\gamma',J)$$

FTC: $\frac{1}{2}\frac{d}{dt}|J(t)|^2 = \underbrace{\frac{1}{2}\frac{d}{dt}|J|^2\Big|_{t=0}}_{= \langle J, J'\rangle|_0 = 0} + \underbrace{\int_0^t |J'|^2 - R(J,\gamma',\gamma',J)\, dt}_{= I_t(J,J)}$

$$= I_t(J,J) \quad (\text{ditto for } \tilde{J})$$

index comparison: $I_t\left(\frac{J}{|J(t)|}, \frac{J}{|J(t)|}\right) \geq \tilde{I}_t\left(\frac{\tilde{J}}{|\tilde{J}(t)|}, \frac{\tilde{J}}{|\tilde{J}(t)|}\right)$

$$\Rightarrow \frac{\frac{d}{dt}|J|^2}{|J|^2} \geq \frac{\frac{d}{dt}|\tilde{J}|^2}{|\tilde{J}|^2}$$

$u = |J(t)|^2$, $v(t) = |\tilde{J}(t)|^2 \Rightarrow \frac{u'}{u} - \frac{v'}{v} \geq 0$

$$\Rightarrow u'v - uv' \geq 0$$

$$\Rightarrow \left(\frac{u}{v}\right)' = \frac{u'v - vu'}{v^2} \geq 0$$

and $u(t) = |J'(0)|^2 t^2 + O(t^3)$
$\qquad\qquad\qquad \shortparallel$
$\quad\ \ v(t) = |\tilde{J}'(0)|^2 t^2 + O(t^3)$ $\Rightarrow \frac{u}{v} \to 1 \quad t \to 0$

$\square$

Cor 1: if $0 < k_- \leq K \leq k_+$, γ = geodesic on M

$\Rightarrow$ dist D between conj pts satisfies

$$\frac{\pi}{\sqrt{k_+}} \le D \le \frac{\pi}{\sqrt{k_-}}$$

proof: $\gamma : [0,\ell] \to M$ geodesic in M, PBAL

spse ℓ = first conj. pt 0, $V(t)$ = Jacobi field st $V(0) = V(\ell) = 0$, $V \perp \gamma$

① $\tilde{\gamma} : [0, \frac{\pi}{\sqrt{k_+}}] \to M_{k_+} = \tilde{M}$

↳ $\tilde{V}$ Jacobi field on $\tilde{\gamma}$ st $\tilde{V}(0) = 0$, $|\tilde{V}'(0)| = |V'(0)|$, $\tilde{V} \perp \tilde{\gamma}$

$\Rightarrow |V(t)| \ge |\tilde{V}(t)| = |V'(0)| \, sn_{k_+}(t)$

↑ non-zero for $t \in (0, \frac{\pi}{\sqrt{k_+}})$

② $\tilde{\gamma} : [0, \frac{\pi}{\sqrt{k_-}}] \to M_{k_-}$

$\cdots \Rightarrow |V(t)| \le |\tilde{V}(t)| = |V'(0)| \, sn_{k_-}(t)$ valid $\forall t \le \ell$

↑ $= 0$ @ $\frac{\pi}{\sqrt{k_-}}$ ∎

Cor 2: $M \ni p$, $\tilde{M} \ni \tilde{p}$ st $K \le \tilde{K}$

spse $\exp_p : B_r \to M$, $\widetilde{\exp}_{\tilde{p}} : B_r(0) \to \tilde{M}$ diffeom onto images

$I : T_pM \to T_{\tilde{p}}\tilde{M}$ = linear isometry

the $f = \widetilde{\exp}_{\tilde{p}} \circ I \circ \exp_p : B_r(p) \to B_r(\tilde{p})$

satisfies $|Df\, w| \le |w|$ "loc dist decr."

$$\begin{tikzcd} T_pM \arrow[r, "I"] \arrow[d, "\exp"] & T_{\tilde p}\tilde M \arrow[d, "\widetilde{\exp}"] \\ M & \tilde M \end{tikzcd}$$

<u>proof:</u> take $v, w \in T_pM$, $|v| = 1$, $w \perp v$

↳ $I(v) \in T_{\tilde p}\tilde M$, $|Iv| = 1$, $Iw \perp Iv$

$$\Rightarrow \underset{V(t)}{\underset{\shortparallel}{D\exp_p\big|_{tv}(tw)}} \quad , \quad \underset{\tilde V(t)}{\underset{\shortparallel}{D\widetilde{\exp}_{\tilde p}\big|_{tIv}(tIw)}} \quad \text{Jacobi fields}$$

$\gamma(t) = \exp_p(tv)$ s.t. $V(0) = \tilde V(0)$, $|V'(0)| = |w| = |\tilde V'(0)|$

$\tilde\gamma(t) = \widetilde{\exp}_{\tilde p}(tIv)$ and $V \perp \gamma$, $\tilde V \perp \tilde\gamma$

$$\text{Rauch} \Rightarrow |D\exp_p\big|_{tv}(w)| \geq |D\widetilde{\exp}_{\tilde p}\big|_{tIv}(Iw)|$$

---

if $w \parallel v$ $\Rightarrow$ $V(t) = t\,w(t)$ ← parallel transport

$\tilde V(t) = t(Iw)(t)$

$$\Rightarrow |D\exp_p\big|_{tv}(w)| = |w| = |D\widetilde{\exp}_{\tilde p}\big|_{tIv}(Iw)|$$

---

if w arbitrary $\Rightarrow$ $v = w^T + w^\perp$ for $w^T \parallel v$

$v^\perp \perp v$

$$\Rightarrow I(w) = I(w^T) + I(w^\perp) \qquad (Iw)^T \parallel Iv$$
$$= I(w)^T + I(w)^\perp \qquad (Iw)^\perp \perp Iv$$

Gauss-lemma: $|D\exp_p|_{tv}(w)|^2$

$$= |D\exp_p|_{tv}(w^T)|^2 + |D\exp_p|_{tv}(w^\perp)|^2$$
$$\geq |D\widetilde{\exp}_{\tilde p}|_{tIv}((Iw)^T)|^2 + |D\widetilde{\exp}_{\tilde p}|_{tIv}(Iv^\perp)|^2$$
$$= |D\widetilde{\exp}_{\tilde p}|_{tIv}(Iv)|^2 .$$

$q = \exp_p(tv)$

$X = D\exp_p|_{tv}(w)$

$$\Rightarrow |X|^2 \geq |Df|_q X|^2$$

$$f = \widetilde{\exp}_{\tilde p} \circ I \circ \exp_p^{-1} \qquad \square$$

Cor 3: if $k_- \leq K \leq k_+$, (x^i) normal coords for $B_r(p)$

$\hookrightarrow x, w \in T_pM$, $|x| < r$

$$\Rightarrow \frac{\operatorname{sn}_{k_+}|x|}{|x|}|w| \leq |D\exp_p|_x(w)| \leq \frac{\operatorname{sn}_{k_-}|x|}{|x|}|w|$$

if $w \perp x$

$(|D\exp_p|_x(w)| = |w|$ if $w \parallel x)$

metric of const curvature

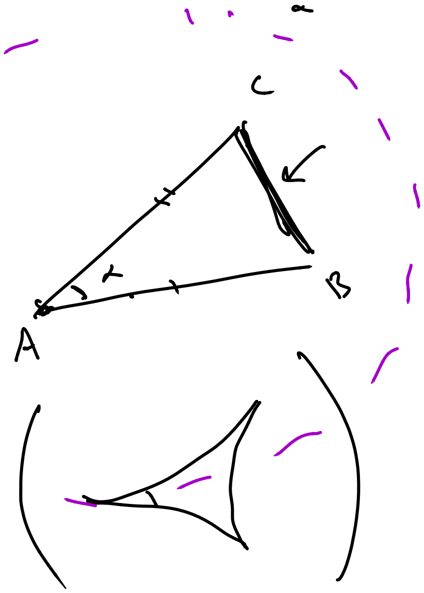
cor 4 (local triangle comparison)

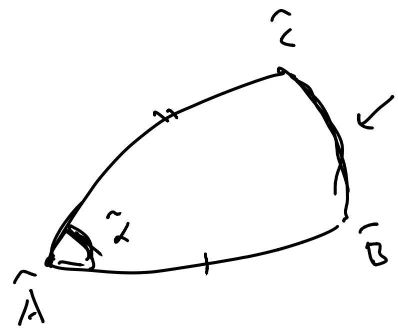
let ABC, $\hat{A}\hat{B}\hat{C}$ be (small) geodesic triangles in $M, \hat{M}$

s.t. $d(A,B) = d(\hat{A},\hat{B})$ and $\underline{K \leq \hat{K}}$

$d(A,C) = d(\hat{A},\hat{C})$ $\Rightarrow$ $d(B,C) \geq d(\hat{B},\hat{C})$

$\underset{\substack{\shortparallel \\ \alpha}}{\measuredangle BC} = \underset{\substack{\shortparallel \\ \hat{\alpha}}}{\measuredangle \hat{B}\hat{C}}$



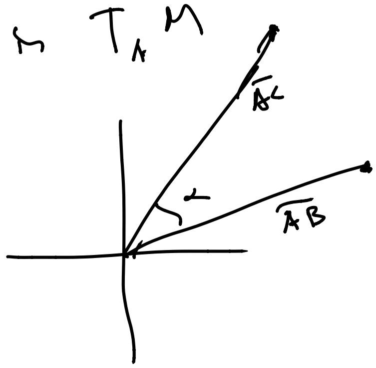


proof: assume that

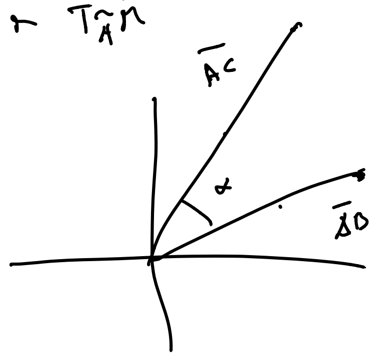
$ABC \subset$ normal coord chart @ A

$\hat{A}\hat{B}\hat{C} \subset$ normal coord chart @ $\hat{A}$

in $T_A M$



in $T_{\hat{A}}\hat{M}$



lin.

can choose isometry $I : T_A M \to T_{\hat{A}}\hat{M}$

s.t. $I(\dot{C}) = \dot{\hat{C}}$, $I(\dot{B}) = \dot{\hat{B}}$

$\hookrightarrow f = \widehat{\exp}_{\hat A} \circ I \circ \exp_A^{-1}$ takes $f(A) = \hat A$
$f(B) = \hat B$
$f(C) = \hat C$

$\gamma(t)$ = min'z geodesic $B \to C$
(parametrizing side $\overline{BC}$)

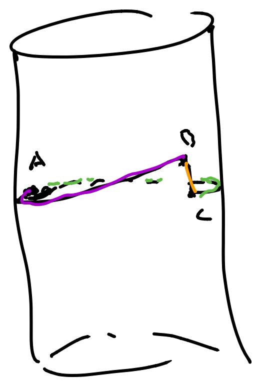
$\to f\circ\gamma$ curve $\hat B \to \hat C$

$\Rightarrow d(\hat B, \hat C) \le L(f\circ\gamma) = \int_0^l |Df\,\gamma'| \le \int_0^l |\gamma'|$

$= L\gamma = d(B,C)$ ∎

Ex: $S^1 \times \mathbb{R}$

$K \equiv 0$

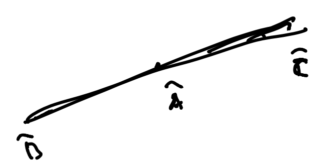


$\overline{AB} = \overline{AC} \approx \pi$

$\measuredangle BC = \pi$

$\overline{BC} \approx \varepsilon$

in $\mathbb{R}^2$:



fails: $K \le 0 = K_{\mathbb{R}^2}$ but $\overline{BC} \not\ge \overline{\hat B \hat C} = 2\pi$

works: $K \ge 0$ and $\overline{BC} \le \overline{\hat B \hat C} = 2\pi$
$\overset{\shortparallel}{\varepsilon}$

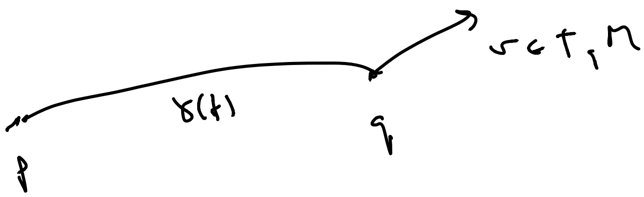
in fact: upper bound on $\overline{BC}$ holds for "large" geodesic triangles

intuition: "topology only shrinks distances" distance Hessian

(M^n, g) complete, $p, q \in M$, $\rho = \text{dist}(\cdot, p)$

assume ρ smooth near q ($\Leftrightarrow q \notin \text{cut}(p)$)

↳ $\gamma(t) = \exp_p(tx) : [0,\ell] \to M$ = (unique) min'z geodesic $p \to q$
PBAL



want to understand: $\nabla^2 \rho|_q (v,v)$

[$d : M \times M \to \mathbb{R}$ = 1-Lipschitz

$|d(p,z) - d(z,q)| \le d(p,q)$ (Δ-ineq.)

$\rho = \text{dist}(\cdot, p) \longrightarrow |\nabla \rho| = 1$ where ρ smooth
i.e. $M \setminus \text{cut}(p)$

recall: $\nabla^2 f(X,Y) = X(Y(f)) - (\nabla_X Y) f$ ⟵ Hessian

$= \langle \nabla_X \nabla f, Y \rangle$

= sym. bilinear form

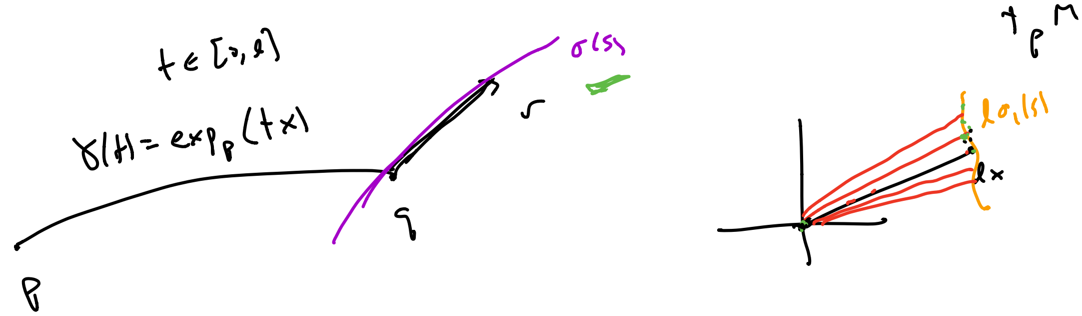
↳ determined by $\nabla^2 f(X,X)$

$$\left[\, 2\nabla^2 f(X,Y) = \nabla^2 f(X+Y, X+Y) - \nabla^2 f(X,X) - \nabla^2 f(Y,Y) \,\right]$$

$\to \sigma(s)$ = geodesic $\Rightarrow \nabla^2 f(\sigma'(0), \sigma'(0))$

$$= \sigma'(\sigma'(f)) - \cancel{(\nabla_{\sigma'}\sigma')} f$$

$$= \frac{d^2}{ds^2} f(\sigma(s))\Big|_{s=0}$$



$\sigma(s) : (-\varepsilon, \varepsilon) \longrightarrow M$ geodesic st. $\sigma(0) = q$, $\sigma'(0) = v$

$$\Rightarrow \nabla^2 \rho|_q(v, v) = \frac{d^2}{ds^2} \rho(\sigma(s))\Big|_{s=0}$$

$(\gamma(\ell) = q = \exp_p(\ell x))$

$\Rightarrow \sigma(s) = \exp_p(\ell \sigma_1(s))$, $\ell\sigma_1(s)$ curve in T_pM

define $\gamma_s(t) = \exp_p(t\, \sigma_1(s))$ $\quad s \in (-\varepsilon, \varepsilon)$, $t \in [0, \ell]$

= family of geodesics $p \longrightarrow \sigma(s)$ minz

and $\gamma_0 = \gamma$

$$\Rightarrow \rho(\sigma(s)) = L\gamma_s$$

$$\Rightarrow \nabla^2 \rho|_q(v, v) = \frac{d^2}{ds^2}\Big|_{s=0} \rho(\sigma(s))$$

$$= \frac{d^2}{ds^2}\Big|_{s=0} L\gamma_s$$

$$= I_\ell(V^\perp, V^\perp) + \underbrace{\langle \nabla_V V, \gamma' \rangle\big|_0^\ell}_{=0}$$

where: $V = \partial_s \gamma_s \big|_{s=0}$, $I(V,W) = \int_0^\ell \langle V', W'\rangle - R(V,\gamma',\gamma',W)\,dt$ along γ

$V^\perp = V - \langle V, \gamma'\rangle \gamma'$

$\hookrightarrow \gamma_s(0) = p \ \forall s \Rightarrow \partial_s \gamma_s \big|_{s=0,t=0} = \nabla_{\partial_s} \partial_s \big|_{t=0, s=0} = 0$

$\Rightarrow \langle \nabla_V V, \gamma' \rangle \big|_{t=0} = 0$

$\hookrightarrow \gamma_s(\ell) = \sigma(s) \Rightarrow \nabla_V V \big|_{t=\ell} = \nabla_{\sigma'} \sigma' = 0$

$\Rightarrow \langle \nabla_V V, \gamma' \rangle \big|_{t=\ell} = 0$

So $\nabla^2 \rho \big|_q (v,v) = I_\ell(W,W)$, W = Jacobi field along γ s.t. $W(0) = 0$, $W(\rho=\ell) = v^\perp$

$= \int_0^\ell |W'|^2 - R(W,\gamma',\gamma',W)\,dt$

$= \langle W(\ell), W'(\ell) \rangle - \int_0^\ell \langle W, \cancel{J W} \rangle^{0}\, dt$

$= \langle W(\ell), W'(\ell) \rangle$

Note: γ = min geodesic $p \to q$, PBAL

$\hookrightarrow \gamma'(t) = \nabla \rho \big|_{\gamma(t)}$

$\Rightarrow v^\perp = v - \langle v, \nabla\rho \rangle \nabla \rho$

Note: $\nabla^2 \rho \big|_q (\nabla \rho, \cdot) = 0$ $\quad \nabla\rho$ = gradient of ρ

$\rightarrow \nabla^2 \rho (v,v) = \nabla^2 \rho (v^\perp, v^\perp)$

$\Big($ if V Jacobi field $V(0)=0$, $V(\rho)=v$
$\Rightarrow V^\perp = V - \langle V,\gamma'\rangle\gamma' =$ (unique) Jacobi field $V^\perp(0)=0$, $V^\perp(\rho)=v^\perp \Big)$

$\Rightarrow$ if $v \parallel \nabla\rho$

then $2\nabla^2\rho(v,w) = \nabla^2\rho(v+w,v+w) - \nabla^2\rho(v,v) - \nabla^2\rho(w,w)$

$= \nabla^2\rho(w^\perp,w^\perp) - \nabla^2\rho(0,0) - \nabla^2\rho(w^\perp,w^\perp)$

$= 0$

special cases: $K = k =$ const, $M = M_k =$ spaceform w/ $K=k$

$p,q \in M_k$, $\gamma:[0,\rho] \to M_k$ min'z $p \to q$ PBAL

(if $k>0$, assume $\rho < \frac{\pi}{\sqrt{k}}$)

$\nabla^2\rho|_q(v,v) = I_\rho(V,V)$, V Jacobi field $V(0)=0$, $V(\rho)=v^\perp$

$\left(\hookrightarrow V(t) = \underset{\uparrow}{v^\perp(t)}\,\dfrac{sn_k(t)}{sn_k(\rho)} \right.$
parallel transport of $v^\perp$ along γ $\Big)$

$= \int_0^\rho |V'|^2 - R(V,\gamma',\gamma',V)\,dt$

$= \langle V(\rho), V'(\rho)\rangle$

$= |v^\perp|^2 \dfrac{sn_k'(\rho)}{sn_k(\rho)}$, $v^\perp = v - \langle v,\nabla\rho\rangle\nabla\rho$

$v^T = \langle v,\nabla\rho\rangle\nabla\rho$

$\underline{k=0}$: $\quad sn_0(t) = t \Rightarrow \nabla^2\rho|_q(v,v) = |v^\perp|^2 \frac{1}{\rho}$

$$\Rightarrow \nabla^2\rho^2|_q(v,v) = 2\rho\, \nabla^2\rho(v,v) + 2\,(d\rho\otimes d\rho)(v,v)$$

$$\left[\ \nabla^2\varphi(f) = \varphi'\,\nabla^2 f + \varphi''\, df\otimes df\right.$$

$$= 2\rho\,\frac{|v^\perp|^2}{\rho} + 2|v^\top|^2$$

$$= 2|v|^2$$

$$\Rightarrow \underline{\nabla^2\rho^2|_q = 2g|_q} \leftarrow \text{metric @ } q$$

$\underline{k=-1}$: $\quad sn_{-1}(t) = \sinh(t) \Rightarrow \nabla^2\rho|_q(v,v) = |v^\perp|^2 \frac{\cosh(\rho)}{\sinh(\rho)}$

$$\Rightarrow \nabla^2\cosh(\rho) = \sinh(\rho)\,\nabla^2\rho + \cosh(\rho)\, d\rho\otimes d\rho$$

$$= \cosh(\rho)\, g$$

$\underline{k=1}$: $\quad sn_1(t) = \sin(t) \Rightarrow \nabla^2\rho(v,v) = |v^\perp|^2 \frac{\cos\rho}{\sin\rho} \qquad \underline{(\rho<\pi)}$

$$\Rightarrow \nabla^2\cos(\rho) = -\sin\rho\,\nabla^2\rho - \cos\rho\, d\rho\otimes d\rho$$

$$= -\cos\rho\, g\,.$$

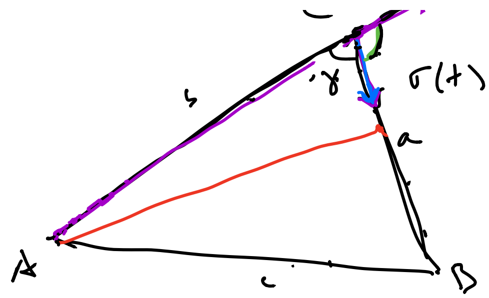
geodesic triangles in space forms

$M_k \quad (k=-1,0,1) \qquad \Delta ABC = \text{geodesic triangle in } M_k$

edges minz.,
avoid cut locus

$\nabla\rho$

r



$\sigma(t)$ params edge a by arclength

$\underline{k=1}$: $\rho = d(\cdot, A)$ smooth on ΔABC

$f(t) = \cos(\rho(\sigma(t)))$

$f(0) = \cos b$, $f(a) = \cos(c)$

$f'' = \nabla^2 \cos(\rho)\,(\sigma', \sigma')$

$= \underbrace{-\cos(\rho)}_{f}\ \underbrace{g(\sigma', \sigma')}_{=1}$

$= -f$

$f'(0) = -\sin b\ \langle \nabla\rho, \sigma'(0)\rangle$

$= -\sin b\ \cos(\pi - \gamma)$

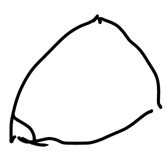
$= \sin b \cos\gamma$

$f'' + f = 0$, $f(0) = \cos b$, $f'(0) = \sin b \cos\gamma$

$\Rightarrow f = \cos b \cos t + \sin b \cos\gamma \sin t$

$\Rightarrow (t=a)$ $\underline{\cos(c) = \cos a \cos b + \sin a \sin b \cos\gamma}$

"cosine law for spherical triangles"



$\underline{k=0}$: $f = \rho(\sigma(t))^2$

$f(0) = b^2$, $f(a) = c^2$

$f'(0) = 2b\,\langle \nabla\rho, \sigma'(0)\rangle$

$= -2b\cos\gamma$

$f'' = \nabla^2\rho^2(\sigma', \sigma')$ $\Rightarrow f(t) = t^2 - 2b\cos\gamma\, t + b^2$

$= 2\, g(\sigma', \sigma')$

$= 2$ $\Rightarrow$ $\underline{c^2 = a^2 + b^2 - 2ab\cos\gamma}$

$\underline{k=-1:}$ $f = \cosh \rho(\sigma(t))$, $f(0) = \cosh b$, $f(c) = \underline{\cosh c}$

$f'(0) = -\sinh b \cos\gamma$

$f'' = \nabla^2 \cosh\rho\, (\sigma', \sigma')$

$= \cosh\rho\; g(\sigma', \sigma')$

$= f$ $\Rightarrow$ $f(t) = \cosh b \cosh t - \sinh b \cos\gamma \sinh t$

$f'' - f = 0$ $\Rightarrow$ $\underline{\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma}$

$\underline{\text{Hessian comparison:}}$ (M, g) complete (?) $K \geq k$ $(\leq k)$

$p, q \in M$, $\rho = \mathrm{dist}(\cdot, p)$, spse ρ smooth near q

① $k = 1$: $\nabla^2 \cos\rho \underset{(\leq)}{\geq} -\cos\rho\; g$, provided $\underline{\rho < \pi}$

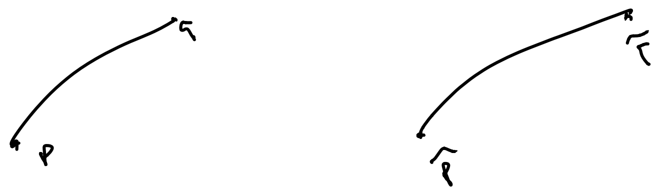
② $k = 0$: $\nabla^2 \rho^2 \underset{(\geq)}{\leq} 2g$

③ $k = -1$: $\nabla^2 \cosh\rho \underset{(\geq)}{\leq} \cosh\rho\; g$

$\underline{\text{proof:}}$ $K \geq 0$. $\tilde{M} = \mathbb{R}^n$

$\gamma(t) = $ mmz $p \to q$ in M, $t \in [0, \rho]$ PBAL

$\tilde{\gamma} = $ mmz $\tilde{p} \to \tilde{q}$ in $\tilde{M}$ w/ $L\tilde{\gamma} = L\gamma = \rho$



$$\nabla^2\rho^2(v,v) = 2\rho\,\nabla^2\rho(v,v) + 2\langle \underset{\shortparallel}{\nabla\rho}, v\rangle^2$$

$$= 2\rho\, I_\rho(V,V) + 2|v^T|^2$$

$$\le 2\rho\, \tilde I_\rho(\tilde V,\tilde V) + 2|v^T|^2 \qquad \text{index for comparison}$$

since $\tilde\nabla^2\tilde\rho = 2\tilde g$
$$= 2\rho\cdot\frac{|v^\perp|^2}{\rho} + 2|v^T|^2 = 2|v|^2$$

I_ρ = index form on $\gamma(t)$, $V(t)$ = Jacobi field on γ st $V(0)=0$, $V(\rho) = v^\perp = v - v^T$

$\tilde I_\rho$ = index form on $\tilde\gamma$, $\tilde V$ = any Jacobi field on $\tilde\gamma$ st $\tilde V(0)=0$, $|\tilde V(\rho)| = |v^\perp|$, $\tilde V \perp \tilde\gamma$ □

<u>ODE comparison:</u> set $L_k f = f'' + kf$

$f, \bar f : [0,\ell] \to \mathbb{R}$ continuous, smooth on $(0,\ell)$

① $(k=1)$ spse $L_1 f \ge a$ $(\le a)$, $\underline{L_1 \bar f = a}$, $\ell < \pi$

and $f(0) \le \bar f(0)$ $(\ge)$, $f(\ell) \le \bar f(\ell)$ $(\ge)$

$\Rightarrow f \le \bar f$ $(\ge)$

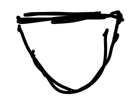
② $(k=0)$ $L_0 f \ge a$, $L_0 \bar f = a$, ...

$\Rightarrow f \le \bar f$ $(\ge)$

③ $(k=-1)$ $L_{-1} f \ge a$, $L_{-1}\bar f = a$, ...

$\Rightarrow f \leq \bar{f}$
$(\geq)$

image: $f'' \geq 0$



<u>proof:</u> $\varphi = f - \bar{f}$

$\Rightarrow L_k \varphi \geq 0$, $\varphi(0) \leq 0$, $\varphi(\ell) \leq 0$

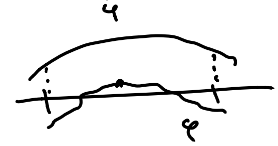
$\hookrightarrow$ WTS: $\varphi \leq 0$

claim: $\exists\ \bar{\varphi} : [0,\ell] \to \mathbb{R}$ smooth s.t. $\bar{\varphi} > \delta > 0$, $L_k \bar{\varphi} < -\delta < 0$

$k = 1$: $\bar{\varphi} = \sin(\alpha t + \beta)$, $\alpha > 1$, $\beta > 0$

$k = 0$: $\bar{\varphi} = -\varepsilon t^2 + \beta$ $\quad \beta > 0$

$k = -1$: $\bar{\varphi} = \cosh(\alpha t)$, $\alpha < 1$



choose least $c \geq 0$ st $c\bar{\varphi} \geq \varphi$ $\quad$ ($c = 0$ ✓)

spse $c > 0$, $\Rightarrow \exists\ t^* \in (0,\ell)$ s.t. $c\bar{\varphi} = \varphi$ @ t^*

$\hookrightarrow$ $\zeta = \varphi - c\bar{\varphi}$ then $\zeta \leq 0$, $L_k \zeta > 0$

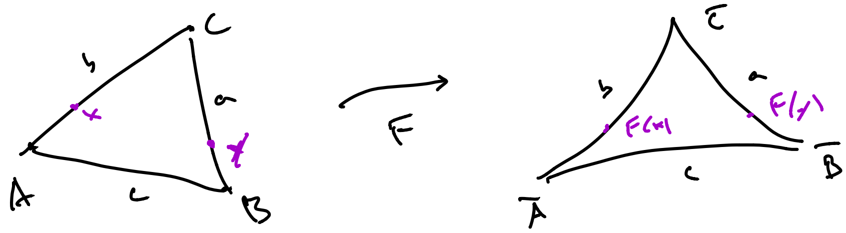
w/ $\zeta(t^*) = 0$

$\Rightarrow 0 \geq \zeta''(t^*) = \zeta'' + k\zeta \big|_{t^*} = L_k \zeta \big|_{t^*} > 0$ $\quad$ ↯ $\quad$ ∎

<u>local Toponogov Δ-comparison</u>: (M, g) complete, $K \geq k$ $(\leq k)$

let ΔABC = (small) geodesic Δ in M
(edges minz)

$\Delta \bar{A}\bar{B}\bar{C}$ = (small) geodesic Δ in M_k = spaceform w/ $K \equiv k$
w/ same side lengths



defn $F : \Delta ABC \to \Delta \bar{A}\bar{B}\bar{C}$ $F(A) = \bar{A}$, etc.

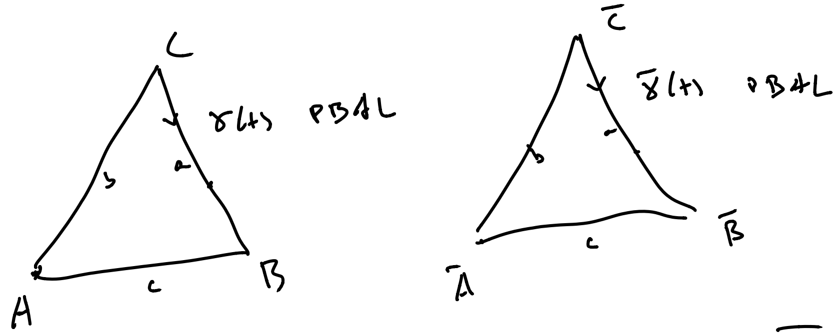
map sides to sides

$\Rightarrow$ F dist decreasing (i.e. $\bar{d}(F(x), F(y)) \leq d(x,y)$)

(resp. increasing)

(if $K > k$ ($< k$) $\Rightarrow$ strictly decr. (incr.))

proof:
$K \geq k = 1$



$\rho = \text{dist}(\cdot, A)$ smooth on $\gamma(t)$, $\bar{\rho} = \overline{\text{dist}}(\cdot, \bar{A})$

$f(t) = \cos\rho\, \gamma(t)$, $\bar{f} = \cos\bar{\rho}\,(\bar{\gamma}(t))$

($\rho < \pi$, $\bar{\rho} < \pi$, $L\gamma < \pi$ since small...)

$$\Rightarrow f'' = \nabla^2 \cos\rho\,(\gamma', \gamma') \geq -\cos\rho\, g(\gamma', \gamma') = f$$

so $f'' + f \geq 0$

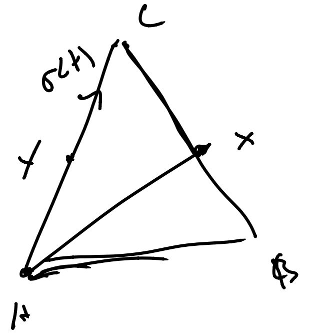
$\bar{f}'' + \bar{f} = 0$

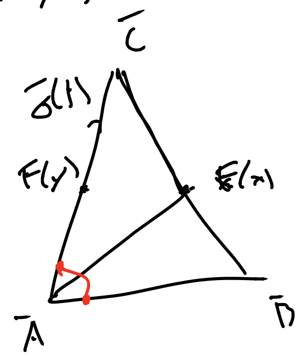
and $f(0) = \cos b = \bar{f}(0)$

$f(a) = \cos c = \bar{f}(a)$

$\Rightarrow$ (ODE) $\cos\rho(\gamma(t)) \leq \cos\bar{\rho}\,\bar{\gamma}(t)$

$\Rightarrow \quad d(\gamma(t), A) \geq \bar{d}(\bar{\gamma}(t), \bar{A})$





$$d(A, x) \geq \bar{d}(\bar{A}, Fx)$$

$$\rho = d(\cdot, x) \quad , \quad \bar{\rho} = \bar{d}(\cdot, F(x))$$

$$f = \cos \rho\, \sigma(t) \quad , \quad \bar{f} = \cos \bar{\rho}\, \bar{\sigma}(t)$$

$$f'' + f \geq 0 \qquad \bar{f}'' + \bar{f} = 0$$

$$f(0) = \cos d(A, x) \leq \cos \bar{d}(\bar{A}, Fx) = \bar{f}(0)$$
$$f(b) = \cos d(C, x) = \cos \bar{d}(\bar{C}, Fx) = \bar{f}(b)$$

$\Rightarrow f \leq \bar{f}$

$\Rightarrow d(\sigma(t), x) \geq \bar{d}(\bar{\sigma}(t), Fx)$ $\square$

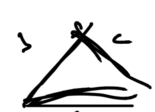
<u>Global Toponogov</u> (M, g) complete, $K \geq k$

ΔABC = geodesic triangle in M, edges min'z

$\Rightarrow \exists$ geodesic triangle $\bar{\Delta}\bar{A}\bar{B}\bar{C}$ in M_k (edges min'z)

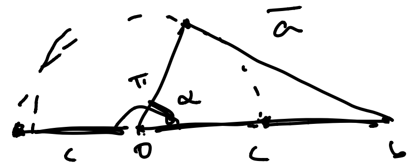
same side length

and $F: \Delta \to \bar{\Delta}$ dist decreasing



<u>existence of $\bar{\Delta}$:</u> $a \geq b \geq c > 0$ sides of Δ

$k = 0, -1$: $\exists\, \bar{\Delta} \subset M_k^2$ w/ same side lengths $\iff a \le b + c$ ← follows by Δ-ineq



open up triangle

$\bar{a}$ continuously goes $b - c \to b + c$

$$\cosh(\bar{a}) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos\alpha$$

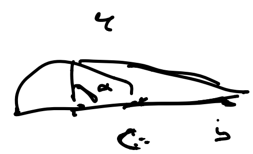
$$\alpha \nearrow \Rightarrow \bar{a} \nearrow$$

$k = 1$: $\exists\, \bar{\Delta} \subset M_k^2$ w/ sides a, b, c (unh z)

$\iff$ ① $a \le b + c$ ← follows by Δ-ineq

② $a \le \pi$ ← follows by Bonnet–Myers

③ $a + b + c \le 2\pi$



$$\cos(\bar{a}) = \cos(b)\cos(c) + \sin(a)\sin(b)\cos\alpha$$

$$\alpha \nearrow \Rightarrow \bar{a} \nearrow$$

min: $\cos(\bar{a}) = \cos b \cos c + \sin a \sin b$

$= \cos(b - c) \Rightarrow \bar{a} = b - c$

max: $\cos(\bar{a}) = \cos b \cos c - \sin a \sin b$

$= \cos(b + c) = \cos(2\pi - (b + c))$

if $b + c \le \pi \Rightarrow \bar{a} = b + c$ $\quad (\Rightarrow \bar{a} + b + c = 2\pi)$

if $b+c > \pi \Rightarrow \underline{\bar{c} = 2\pi - (b+c)}$

<u>aside:</u> if $a = \pi$

$\Downarrow$

$\cos(c) = -\cos(b)$

$\bar{\Delta}$ not unique

<u>claim:</u> if $a \geq b \geq c > 0$, $\underline{a \leq b+c}$, $a \leq \pi$

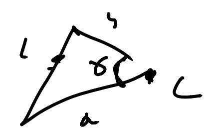
then $\underline{a+b+c \leq 2\pi} \iff \exists\, \bar{f} : [0,a] \to [-1,1]$

st $\bar{f}'' + \bar{f} = 0$

$\bar{f}(0) = \cos b$, $\bar{f}(a) = \cos(c)$

(and $\bar{f} \geq -1$)

<u>proof of claim:</u> if $b \leq \pi/2$ ✓



spse $b > \pi/2$

↳ $\bar{f}(t) = \underbrace{\underset{<0}{\cos b}\,\cos t}\; + \underbrace{\underset{<0}{G}\,\underset{\geq 0}{\sin b}\,\sin t}$ $\quad G = \frac{\cos c - \cos b \cos a}{\sin b \sin a} = \cos\gamma$

$c \leq 2\pi - a - b \iff \underline{G \geq -1}$

$c \geq a - b \iff \underline{G \leq 1}$

NTS: $G \geq -1$ $\quad \Rightarrow \min\limits_{[0,a]} \bar{f} = \min\limits_{\mathbb{R}} \bar{f}$ occurs when $\cos t > 0$, $\sin t > 0$

spse $G < 0$

$f'(t_0) = 0 \Rightarrow \tan t_0 = G \tan b$

$\Rightarrow f(t_0) = \pm \sqrt{\frac{1 + G^2 \tan^2 b}{1 + \tan^2 b}}$

$\geq -1 \Rightarrow |G| < 1$ $\quad \square$

weak solns: $L_k f = f'' + kf$

$f : [0,\ell] \to \mathbb{R}$ = weak sub (super) soln of $L_k f = a$

$L_k f \,\tilde{\geq}\, a$ $\qquad$ $L_k f \,\tilde{\leq}\, a$ $\qquad$ "viscosity soln"

$\iff$ f continuous, $\forall\, t_0 \in (0,\ell)$, $\forall\, \varepsilon > 0$

$\exists$ interval $I_{t_0} \ni t_0$, smooth fn $f_{t_0,\varepsilon} : I_{t_0} \to \mathbb{R}$

st. $f_{t_0,\varepsilon} \leq f$, $f_{t_0,\varepsilon}(t_0) = f(t_0)$

and $L_k f_{t_0,\varepsilon} \geq a - \varepsilon$

(super: reverse inequalities, $a + \varepsilon$, etc.)

eg. $f'' \geq 0$, $f'' \,\tilde{\geq}\, 0$

weak ODE comparison: $f, \bar{f} : [0,\ell] \to \mathbb{R}$ continuous

$\bar{f}$ smooth soln to $L_k \bar{f} = a$

and $f(0) \leq \bar{f}(0)$ $(\geq)$, $f(\ell) \leq \bar{f}(\ell)$ $(\geq)$

① $(k=1)$ $L_1 f \,\tilde{\geq}\, a$ $(\tilde{\leq})$, $\ell < \pi$ $\Rightarrow$ $f \leq \bar{f}$ $(\geq)$

② $(k = 0, -1)$ $L_k f \,\tilde{\geq}\, a$ $(\tilde{\leq})$ $\Rightarrow$ $f \leq \bar{f}$ $(\geq)$

proof: as before $\exists\, \bar{\varphi} : [0,\ell] \to \mathbb{R}$ st $\bar{\varphi} \geq \delta > 0$, $L_k \bar{\varphi} \leq -\delta < 0$

$\zeta = f - \bar{f} - c\tilde{e}$ st $\zeta \leq 0$, $\zeta(t_0) = 0$, $c > 0$

$\Rightarrow L_u \zeta \mathrel{\tilde{\geq}} c\delta > 0$

$\Rightarrow \exists\, \bar{\zeta} \leq \zeta \leq 0$ st $\bar{\zeta}(t_0) = \zeta(t_0) = 0$

and $L_u \bar{\zeta} \geq c\delta - \varepsilon > 0$

$\Rightarrow 0 \geq \bar{\zeta}''(t_0) = L_u \bar{\zeta} > 0$ $\lightning$ $\square$

<u>weak Hessian comparison</u>:

$f : M \to \mathbb{R}$ continuous then $\nabla^2 f\big|_q \mathrel{\tilde{\geq}} a\, g\big|_q$

(defn)

$\Leftrightarrow \forall \varepsilon\ \exists\, f_{q,\varepsilon}$ defined, smooth near q

st $f_{q,\varepsilon} \leq f$, $= f$ @ q

and $\nabla^2 f \geq (a-\varepsilon) g$